05 Oscillations

Syllabus

5.1 Introdiction

i. Oscillatory motion is a Periodic motion

ii. Displacement, velocity and acceleration in oscillations are represented by sine or cosine functions.

These functions are known as harmonic functions

iii. An oscillatory motion obeying harmonic function is called harmonic motion.

5.2 Explanation for Periodic Motion

Q.1 Explain periodic motion with examples.

Ans: Periodic motion :

i. Any motion which repeats itself in equal intervals of time is called periodic motion.

ii. A particle performing periodic motion goes on repeating the same set of movements.

iii. The time taken for one such set of movements is called the period or periodic time.

Examples :

a. Uniform circular motion of any object is a periodic motion.
(e.g., motion of the electron around nucleus)

b. Rotational motion of any body with constant angular velocity is a periodic motion. (e.g., motion of the second hand of clock)

c. The oscillatory motion is a periodic motion. (e.g., motion of pendulum of clock)

Q.2 Define oscillatory or Vibratory motion.

Ans :

i. Periodic motion in which a particle repeatedly moves to and from along the same path is the ocillatory or vibratory motion

Examples :

a. Motion of Pendulum of Clock.

b. Swinging of a Swing.

Note :

Every oscillatory motion is periodic but every periodic motion need not to be oscillatory.

5.3 Linear Simple Harmonic Motion (S.H.M)

Q.3 Define linear SHM. Give its examples.

Ans: Linear SHM : The linear simple harmonic motion is defined as the linear periodic motion of a body in which the restoring force is always directed towards the mean position and its magnitude is proportional to the displacement from the mean position.

$F \propto -x$

$\therefore$ $F = -kx$

Where k is force constant

Examples:

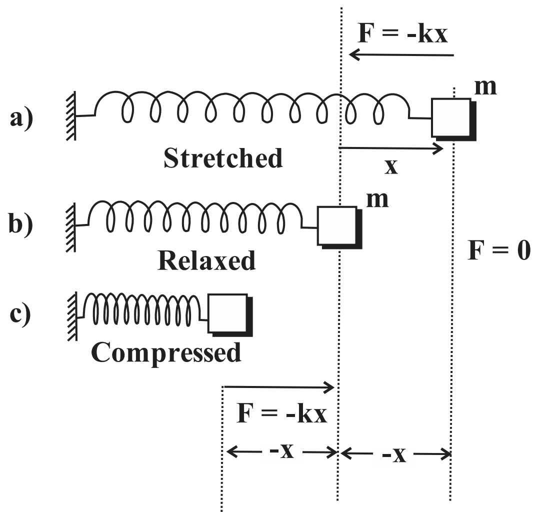
i. Simple pendulum performs linear S.H.M. when angular displacement is small

ii. Needle of sewing machine performs linear S.H.M.

Q.4 Define restoring force. Explain the concept of restoring force in spring mass oscillator.

Ans: The force acting on the particle to bring it in mean position, is known as restoring force.

Explanation:



i. Consider a body of mass m attached to an ideal spring of force constant k and made free to move over a frictionless horizontal surface.

ii. In equilibrium position spring exerts no force on body as shown in figure (b).

iii. If the body is displaced towards right the force exerted by the spring on the body is directed towards the left as shown in figure (a).

iv. If x is displacement due to external force, the restoring force is given by $F = -kx$. Negative sign indicates that the direction of restoring force is opposite to displacement.

5.4 Differential Equation of S.H.M.

Q.5 Obtain the differential equation of linear SHM.

Ans: Differential equation of SHM :

i. In linear SHM,

a. Restoring force is always directed towards the mean position.

b. The magnitude of Restoring force is always directly proportional to the displacement from the mean position.

ii. These two conditions are mathematically expressed as:

$\therefore$ $F \propto -x$

$\therefore$ $F = -kx$ ...(1)

where, k = force constant (i.e., k is the force per unit displacement)

The negative sign indicates that the force and displacement are opposite to each other.

iii. Velocity is rate of change of displacement w.r.t time

$\therefore$ $v = \frac{dx}{dt}$ ...(2)

vi. Accleration is rate of change of velocity w.r.t time

$\therefore$ $a = \frac{dv}{dt}$ $\quad \therefore a = \frac{d}{dt}\left(\frac{dx}{dt}\right)$ ...(From 2)

$a = \frac{d^2x}{dt^2}$ ...(3)

iv. We know that,

$F = ma$

$F = m\frac{d^2x}{dt^2}$ ...(From 3)

Substituting in equation (1) we get

v. $m.\frac{d^2x}{dt^2} = -kx$

$\therefore$ $m.\frac{d^2x}{dt^2} + kx = 0$

$\therefore$ $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

Put, $\frac{k}{m} = \omega^2$,

where, ω is a constant.

$\therefore$ $\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$

This is the differential equation of linear SHM.

Note :

Force constant/spring constant

i. *Force constant is defined as force per unit displacement.*

ii. $k = \frac{F}{x}$

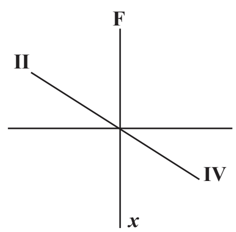
iii. *SI Unit :* $\frac{N}{m}$

CGS unit : $\frac{dyne}{cm}$

iv. *Dimension :* $[M^1L^6T^{-2}]$

v. *K is also calculates as,* $K = m\omega^2$

vi. *Graph of* $\frac{F}{x}$ *is a straight line passing From (ii) and (iv) quadrant slope = – K*



Type - I

Numerical based on differential equation

Formulae used

1. $\frac{d^2x}{dt^2} + \omega^2 x = 0$
2. $\omega^2 = \frac{k}{m}$
3. $T = \frac{2\pi}{\omega}$
4. $n = \frac{1}{T}$
5. $\omega = 2\pi n$
6. $F = -kx$

★ 1) In SI units, the differential equation of an S.H.M. is $\frac{d^2x}{dt^2} = -36x$. Find its frequency and period.

Data: $\frac{d^2x}{dt^2} = -36x$

To Find : i. n ii. T

Formula : i. $\frac{d^2x}{dt^2} + \omega^2 x = 0$ ii. $\omega = 2\pi n$

iii. $T = \frac{1}{n}$

Solution:

i. $\frac{d^2x}{dt^2} + 36x = 0$

Comparing with differential equation,

$\frac{d^2x}{dt^2} + \omega^2 x = 0$

$\therefore$ $\omega^2 = 36$

$\therefore$ $\omega = 6$ rad/s

ii. $\omega = 2\pi n$

$\therefore$ $n = \frac{\omega}{2\pi} = \frac{6}{2\pi} = \frac{3}{3.142} = 0.955\text{Hz}$

iii. Period (T) $= \frac{1}{n} = \frac{1}{0.955} = 1.05\text{s}$

Ans : The frequency and period of SHM are 0.955 Hz and 1.05s respectively.

Problem For Practice

1. The differential equation of particle is $\frac{d^2x}{dt^2} + 64x = 0$ find time period of particle

Ans : 0.785 sec

2. What is the value of frequency if differential equation of SHM is $\frac{d^2x}{dt^2} + 100x = 0$

Ans : 1.592 Hz

5.5 Acceleration (a), Velocity (v) and Displacement (x) of S.H.M.

Q.7 State the differential equation of linear SHM and obtain expressions for acceleration, velocity and displacement in SHM.

Ans:

i. Acceleration in SHM

The differential equation of a linear SHM is,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad ...(1)$$

$\therefore \quad \frac{d^2x}{dt^2} = -\omega^2 x$

$\therefore \quad \boxed{a = -\omega^2 x}$ ...(2)

The negative sign indicates that the direction of acceleration is always opposite to that of the displacement .

Case-1: At mean position

$x = 0$

$\therefore \quad \boxed{a = 0}$

Case-2: At Extreme position

$x = \pm A$

$\therefore \quad \boxed{a_{max} = \mp \omega^2 A}$

ii. **Velocity in SHM**

Acceleration of a particle is ,

$a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = v.\frac{dv}{dx}$

$\therefore$ equation (2) becomes,

$v.\frac{dv}{dx} = -\omega^2 x$

$v.dv = -\omega^2 x. dx$

Integrating both sides, we get,

$\int vdv = \int (-\omega^2 x)dx$

$\therefore \quad \frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c$ ...(3)

where, c is the constant of integration.

Let, the maximum displacement of the particle be A.

When $x = A$, $v = 0$.

Substituting in eq. (3)

$\therefore \quad 0 = -\frac{\omega^2 A^2}{2} + c$

$\therefore \quad c = \frac{\omega^2 A^2}{2}$

$\therefore$ On substitution, equation (3) becomes,

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 A^2}{2}$

$\therefore \quad v^2 = \omega^2 (A^2 - x^2)$

$\therefore \quad \boxed{v = \pm \omega \sqrt{A^2 - x^2}}$ ...(4)

This equation gives the velocity of the particle in terms of its displacement x.This velocity can be +ve or -ve depending upon the direction in which the particle is moving.

Case-1: At mean position

$x = 0$

$\therefore \quad v = \pm \omega \sqrt{A^2 - 0^2}$

$\therefore \quad \boxed{v_{max} = \pm \omega A}$

Case-2: At Extreme position

$x = A$

$\therefore \quad v = \pm \omega \sqrt{A^2 - A^2}$

$\therefore \quad \boxed{v = 0}$

iii. **Accleration in SHM**

For forward motion $v = \omega \sqrt{A^2 - x^2}$

Substituting, $v = \frac{dx}{dt}$, we get,

$\frac{dx}{dt} = \omega \sqrt{A^2 - x^2}$

$\therefore \quad \frac{dx}{\sqrt{A^2 - x^2}} = \omega \, dt$

Integrating both sides,

$\int \frac{dx}{\sqrt{A^2 - x^2}} = \int \omega dt$

$\therefore \quad \sin^{-1}\left(\frac{x}{A}\right) = \omega t + \phi,$

where, ϕ is the constant of integration,

$\therefore \quad \frac{x}{A} = \sin(\omega t + \phi)$

$\therefore \quad \boxed{x = A \sin (\omega t + \phi)}$ ...(5)

This is general expression for the displacement in SHM.

Case-1: When particle starts from mean position

$t = 0; x = 0$

$\therefore \quad 0 = A \sin (\omega \times 0 + \phi)$

$\therefore \quad \sin \phi = 0$

$\therefore \quad \phi = \sin^{-1}(0)$

$\therefore \quad \boxed{\phi = 0}$

Case-2: When particle starts from extreme position

$t = 0; x = A$

$A = A \sin (\omega \times 0 + \phi)$

$\sin \phi = 1$

$\phi = \sin^{-1}(1)$

$\boxed{\phi = 90°}$

Q.8 Explain the terms :

i. Amplitude of SHM

ii. **Phase of SHM**
iii. **Epoch of SHM**

Ans:

1. Amplitude of SHM :

i. The maximum displacement of a particle performing SHM from its mean position is called the amplitude of SHM.

ii. It is denoted by A.

iii. $A = \frac{\text{path of SHM}}{2}$ or

$\therefore$ Path of SHM = 2 A

iv. Distance covered by particle in one oscillation is, d = 4A

2. Phase of SHM :

i. A quantity which describes the state of oscillation i.e. position and direction of vibration of a particle performing SHM is called phase of SHM.

ii. The general equation of displacement in SHM is, $x = A\sin(\omega t + \phi)$.

iii. In this expression, the angle $(\omega t + \phi)$ is called phase of SHM.

3. Epoch of SHM :

i. The initial phase of SHM is called epoch of SHM.

ii. The general equation of displacement in SHM is, $x = A \sin\ (\omega t + \phi)$

$\therefore$ Initial phase of SHM = $(\omega \times 0 + \phi) = \phi$

iii. Thus, ϕ is the initial phase or starting phase or epoch of SHM.

Q.9 Define and obtain expressions for :

i. Period of SHM

ii. Frequency of SHM

Ans:

i. Period of SHM :

The time taken by the particle performing SHM to complete one oscillation is called period of SHM.

Expression for period of SHM :

Consider a particle performing SHM. At any instant t, the displacement x of the particle from the mean position is given by the equation,

$x = A \sin\ (\omega t + \phi)$

where, ω and ϕ are constants.

After time, $t = \frac{2\pi}{\omega}$, the displacement of the particle is,

$$x = A\sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$$

$= A \sin\ [\ \omega t + 2\pi + \phi\ ]$

$= A \sin\ [\ \omega t + \phi\ ] = x$

This shows that, the particle is at the same position after time $\frac{2\pi}{\omega}$.

Also, particle completes one oscillation in time $\frac{2\pi}{\omega}$, and it is moving in the same direction.

$\therefore$ Period of SHM is, $T = \frac{2\pi}{\omega}$

ii. Frequency of SHM :

The number of oscillations completed by the particle performing SHM per unit time is called frequency of SHM.

$\therefore$ $n = \frac{1}{T} = \frac{\omega}{2\pi}$

$\therefore$ $\omega = 2\pi n$

Key Point

Type - II
Numerical based on accleration velocity and displacement

Formula used

1. $F = -kx$

2. $k = \frac{F}{x}$, $k = m\omega^2$

3. $a = -\omega^2 x$
At mean position
$a = 0$
At extreme position
$a_{max} = \mp\omega^2 A$

4. $v = \omega\sqrt{A^2 - x^2}$
At mean position
$v_{max} = \omega A$
At extreme position

v = 0

5. $x = A\sin(\omega t + \phi)$

$v = A\omega\cos(\omega t + \phi)$

6. Phase = $\omega t + \phi$

7. Epoch = ϕ

8. $\omega = \dfrac{2\pi}{T}$

9. $\omega = 2\pi n$

★ 1) A particle performs linear S.H.M. of period 4 seconds and amplitude 4 cm. Find the time taken by it to travel a distance of 1 cm from the positive extreme position.

Data: T = 4s, A = 4 cm = 0.04m,
x = 4 – 1 = 3 cm = 0.03 m
As particle starts from positive extreme position, $\phi = \dfrac{\pi}{2}$

To find: t

Formula: $x = A\sin(\omega t + \phi)$

Solution: $x = A\sin(\omega t + \phi)$

$$x = A\sin\left(\frac{2\pi t}{T} + \phi\right)$$

$$\therefore \quad 3 = 4\sin\left(\frac{2\pi t}{T} + \frac{\pi}{2}\right)$$

$$\therefore \quad \cos\left(\frac{2\pi}{4}\right)t = \frac{3}{4} \ ...\left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$$

$$\therefore \quad \left(\frac{\pi}{2}\right)t = \cos^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \quad \frac{\pi t}{2} = \cos^{-1}(0.75)$$

$$\frac{\pi t}{2} = 41°24' = 41°\left(\frac{24}{60}\right)° = 41.4°$$

$$\frac{\pi t}{2} = 41.4 \times \frac{\pi}{180}$$

$$\therefore \quad = \frac{41.4}{90}$$

t = 0.46 sec

Ans : Time taken by particle to travel a distance of 1 cm from the positive extreme position is 0.46 s.

★ 2) A particle performing linear S.H.M. with period 6 second is at the positive extreme position at t = 0. The particle is found to be at a distance of 3 cm from this position at time t = 7 s, before reaching the mean position. Find the amplitude of S.H.M.

Data: $x = A - 3$, t = 7s, T = 6 sec
Particle starts from positive extreme position

$$\therefore \quad \phi = \frac{\pi}{2}$$

To find: A

Formula: $x = A\sin(\omega t + \phi)$

Solution:

i. $x = A\sin(\omega t + \phi)$

$$A - 3 = A\sin\left(\frac{2\pi}{T}t + \frac{\pi}{2}\right) \ ...\left(\because \omega = \frac{2\pi}{T}\right)$$

$$\therefore \quad \frac{A-3}{A} = \sin\left(\frac{2\pi}{6} \times 7 + \frac{\pi}{2}\right)$$

$$\therefore \quad \frac{A-3}{A} = \sin\left(\frac{7\pi}{3} + \frac{\pi}{2}\right) = \cos\frac{7\pi}{3}$$

$$\left(\because \sin\frac{\pi}{2} + \theta = \cos\theta\right)$$

$$\frac{A-3}{A} = \cos\left(2\pi + \frac{\pi}{3}\right)$$

$$\therefore \quad \frac{A-3}{A} = \cos\frac{\pi}{3} \ ...[\because \cos(2\pi + \theta) = \cos\theta]$$

$$\therefore \quad \frac{A-3}{A} = \frac{1}{2}$$

$\therefore$ 2A – 6 = A

$\therefore$ A = 6 cm

Ans : The amplitude of S.H.M. is 6 cm.

★ 3) The speeds of a particle performing linear S.H.M. are 8 cm/s and 6 cm/s at respective displaements of 6 cm and 8 cm. Find its period and amplitude.

Data: v_1 = 8 cm/s when x_1 = 6 cm
v_2 = 6 cm/s when x_2 = 8 cm

To find: i. Amplitude (A)
ii. Time period (T)

Formulae: i. $v = \omega\sqrt{A^2 - x^2}$ ii. $\omega = \frac{2\pi}{T}$

Solution:

i. $v_1 = \omega\sqrt{A^2 - x_1^2}$ ...(1)

$v_2 = \omega\sqrt{A^2 - x_2^2}$ ...(2)

Dividing equation (1) by equation (2),

$$\frac{v_1}{v_2} = \frac{\not\omega\sqrt{A^2 - x_1^2}}{\not\omega\sqrt{A^2 - x_2^2}}$$

$\therefore$ $\frac{8}{6} = \frac{\sqrt{A^2 - 6^2}}{\sqrt{A^2 - 8^2}} = \frac{4}{3}$

$\therefore$ $\frac{16}{9} = \frac{A^2 - 36}{A^2 - 64}$

$\therefore$ $16A^2 - 1024 = 9A^2 - 324$

$\therefore$ $7A^2 = 700$

$\therefore$ $A^2 = 100$

$\therefore$ $A = 10cm$

ii. $v_1 = \omega\sqrt{A^2 - x_1^2}$

$\therefore$ $8 = \omega\sqrt{10^2 - 6^2} = \omega\sqrt{64} = \omega \times 8$

$\therefore$ $\omega = 1 rad/s$

iii. $T = \frac{2\pi}{\omega} = \frac{2 \times 3.142}{1} = 6.284s$

Ans : The amplitude and time period of oscillation are respectively 10 cm and 6.284 s

★ **4) The maximum velocity of a particle performing S.H.M is 6.28 cm/s. If the length of its path is 8 cm, calculate its period.**

Data: $v_{max} = 6.28$ cm/s $= 2\pi$ cm/s;

$2A = 8cm$ $\therefore$ $A = 4cm$

To find: T

Formulae: i. $v_{max} = A\omega$ ii. $\omega = \frac{2\pi}{T}$

Solution: $v_{max} = A\omega$

$\therefore$ $v_{max} = A\left(\frac{2\pi}{T}\right)$ ... $\left(\because \omega = \frac{2\pi}{T}\right)$

$\therefore$ $2\pi = 4\left(\frac{2\pi}{T}\right)$

$\therefore$ $T = 4$ s

Ans : Time period of oscillatiion of particle is 4 s.

★ **5) The maximum speed of a particle performing linear S.H.M. is 0.08 m/s. If its maximum acceleration is 0.32 m/s², calculate its**

i. period and ii. amplitude.

Data: $v_{max} = 0.08$ m/s, $a_{max} = 0.32$ m/s²

To find: i. T ii. A

Formulae: i. $v_{max} = \omega A$ ii. $a_{max} = \omega^2 A$

iii. $\omega = \frac{2\pi}{T}$

Solution:

i. $v_{max} = \omega A$ and $a_{max} = \omega^2 A$

$\therefore$ $\frac{a_{max}}{v_{max}} = \frac{\omega^2 A}{\omega A} = \omega$

$\therefore$ $\omega = \frac{0.32}{0.08} = 4$ rad/s

ii. $T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} = \frac{3.142}{2} = 1.571s$

iii. $v_{max} = \omega A$

$\therefore$ $0.08 = 4 \times A$

$\therefore$ $A = 0.02$ m = 2cm

Ans : i. Period of oscillation of particle is 1.571 s. and Amplitude is 2 cm.

★ **6) At what distance from the mean position is the speed of a particle performing S.H.M. half is maximum speed. Given: path length of S.H.M. = 10 cm.**

Data: $2A = 10$ cm, $v = \frac{v_{max}}{2}$

$\therefore$ $A = 5$ cm ;

To find: x

Formulae: i. $v_{max} = \omega A$ ii. $v = \omega\sqrt{A^2 - x^2}$

Solution: According to given condition

$v = \frac{v_{max}}{2}$

As $v = \omega\sqrt{A^2 - x^2}$ and $v_{max} = \omega A$

$\therefore$ $\not\omega\sqrt{A^2 - x^2} = \frac{\not\omega A}{2}$

Squaring on both the sides

$\therefore \quad A^2 - x^2 = \frac{A^2}{4}$

$\therefore \quad x^2 = \frac{3A^2}{4}$

$\therefore \quad x = \pm\frac{\sqrt{3}}{2} \times A = \pm\frac{\sqrt{3}}{2} \times 5 = \pm 4.33 \text{ cm}$

Ans :	The distance at which speed of particle is half of its maximum value is 4.33cm.

★ 7) A needle of a sewing moves along a path of amplitude 4 cm with frequency 5 Hz. Find its acceleration $\frac{1}{30}$ s after it has crosed the mean position.

Data: $A = 4$ cm $= 0.04$m, $n = 5$Hz, $t = \frac{1}{30}$ sec

To find: a

Formulae: i. $\omega = 2\pi n$ ii. $a = -\omega^2 A \sin\omega t$

Solution:

i. $\omega = 2\pi n, = 2\pi \times 5 = 10\pi$ rad/s

ii. $|a| = \omega^2 A \sin\omega t$

$|a| = (10\pi)^2 \times 0.04 \times \sin\left(10\pi \times \frac{1}{30}\right)$

$= 100 \times (3.14)^2 \times 0.04 \times \sin\frac{\pi}{3}$

$= 4 \times 9.872 \times \frac{\sqrt{3}}{2} = 2 \times 9.872 \times 1.732$

$= 34.19 \text{ m/s}^2$

Ans :	The magnitude of acceleration at the required instant is 34.19 m/s^2

★ 8) A particle performing linear S.H.M. of period 2π seconds about the mean position O is observed to have a speed of $b\sqrt{3}$ m/s, when at a distance b (meter) from O. If the particle is moving away from O at that instant, find the time required by the particle, to travel a further distance b.

Data: $T = 2\pi$ s, $v = b\sqrt{3}$ m/s, $x = b$ m

To find: Time required by object to travel distance from b to 2b (t)

Formulae: i. $\omega = \frac{2\pi}{T}$ ii. $v = \omega\sqrt{A^2 - x^2}$

iii. $v = \omega A \cos\omega t$ iv. $x = A \sin \omega t$

Solution:

i. $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \text{ rad s}^{-1}$

ii. $v = \omega\sqrt{A^2 - x^2}$

$b\sqrt{3} = (1) \times \sqrt{A^2 - b^2}$

Spuaring both sides,

$3b^2 = A^2 - b^2$

$\therefore \quad 4b^2 = A^2$

$\therefore \quad A = 2b$

iii. Time taken to cover distance $b(t_1)$

$x = A \sin \omega t \ (1 \times t_1)$

$b = 2b \sin$

$\therefore \quad 2b \times \cos$

$\therefore \quad \sin t_1 = \frac{1}{2}$

$\therefore \quad t_1 = \sin^{-1}\left(\frac{1}{2}\right)$

$\therefore \quad t_1 = \frac{\pi}{6}$ sec

iv. Time taken to cover distance $2b(t_2)$

$x = A \sin \omega t$

$2b = 2b \sin (1 \times t_2)$

$1 = \sin t_2$

$t_2 = \sin^{-1} (1)$

$t_2 = \frac{\pi}{2}$

v. Time taken to travel from b to 2b

$t = t_2 - t_1 = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$ s

Ans :	The time required by object is $\frac{\pi}{3}$ s.

★ 9) A body of mass 0.2 kg performs linear S.H.M. It experiences a restoring force of 0.2 N when its displacement from the mean position is 4 cm. Determine

i. force constant

ii. period of S.H.M. and

iii. acceleration of the body when its displacement from the mean position

is 1 cm.

Data: m = 0.2 kg, F = 0.2N,

x_1 = 4cm = 0.04m,

x_2 = 1cm = 0.01 m

To find: i. k ii. T iii. a

Formulae: i. $k = \frac{F}{x}$ ii. $T = \frac{2\pi}{\omega}$

iii. $\omega = \sqrt{\frac{k}{m}}$ iv. $a = -\omega^2 x$

Solution:

i. $k = \frac{F}{x} = \frac{0.2}{0.04} = \frac{20}{4} = 5N/m$

ii. $T = \frac{2\pi}{\omega} = \frac{2\pi}{5} = 0.4\pi\ s$

iii. $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{0.2}} = \sqrt{25} = 5rad/s$

iv. $a = -\omega^2 x$,
for x_2 = 0.01 m,
$a = -5^2 \times 0.01 = -25\times0.01 = -0.25ms^{-2}$

Ans : i. The force constant is 5N/m.
ii. The time period of SHM is 0.4 πs.
iii. Acceleration at x = 1 cm is -0.25 m s^{-2}.

Problem for Practice

1. A simple harmonic motion is represented by x = 10 sin (20t + 0.5). Write down its amplitude, angular frequency, frequency time period and initial phase, if displacement is measured in meteres and time in seconds.

Ans : 10m, 20 rad s^{-1}, 3.18Hz, 0.314s, 0.5 rad

2. A body oscillates with SHM according to the equation $x(t) = 5\cos(2\pi t + \pi/4)$ where t is sec. and x in metres. Calculate (a) Displacement at t = 0 (b) Time period (c) Initial velocity

Ans: (a) $\frac{5}{\sqrt{2}}$ (b)1s (c) $-\frac{10\pi}{\sqrt{2}}$ m / s

3. For a particle in SHM, the displacement x of the particle as a function of time t is given as $x = A\sin(2\pi t)$. Here x is in cm and t is in seconds. Let the time taken by the particle to travel from $x = 0$ to $x = A/2$ be t_1 and the time taken to travel from $x = A/2$ to $x = A$ be t_2. Find t_1/t_2

Ans : $\frac{1}{2}$

4. A particle is moving with SHM in a straight line. When the distance of the particle from the equilibrium position has values x1 and x2, the corresponding values of velocities are u1 and u2. Show that the time period of oscilliation is given by $T = 2\pi\left[\frac{x_2^2 - x_1^2)}{u_1^2 - u_2^2}\right]^{\frac{1}{2}}$

5. The maximum velocity of a particle performing linear S.H.M is 0.16 m/s. If is maximum acceleration is 0.64 m/s^2, calculate its period.

Ans: 1.57 sec

6. A particle in S.H.M. has a period of 2 seconds and amplitude of 10 cm. Calculate the acceleration when it is at 4 cm from its positive extreme position. (Take $\pi^2 = 10$)

Ans : 60 cm/s²

7. The periodic time of a linear harmonic oscillator is 2π second, with maximum displacement of 1 cm. If the particle starts from extreme position, find the displacement of the particle after $\frac{\pi}{3}$ seconds.

Ans: 0.866 cm

8. A particle performing linear S.H.M. has a period of 6.28 seconds and a pathlength of 20 cm. What is the velocity when its displacement is 6 cm from mean position?

Ans : 8 cm/s

9. A particle performing linear S.H.M. has maximum velocity of 25 cm/s and maximum acceleration of 100 cm/s^2. Find the amplitude and period of oscillation (π=3.142)

Ans : 1.57 dec, 6.25 m

5.6 Amplitude, Period and Frequency of S.H.M.

Time period and frequency

i. We know that,

$$\omega^2 = \frac{k}{m} = \frac{\text{force per unit displacement}}{\text{mass}}$$

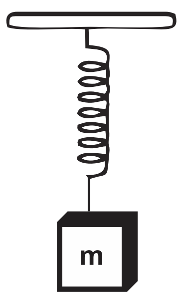
$$\omega^2 = \text{acceleration per unit displacement}$$

$$\omega = \sqrt{\text{acceleration per unit displacement}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\text{acceleration per unit displacement}}}$$

ii. Spring Pendulum

A point mass suspended from a mass less spring or placed on a frictionless horizontal plane attached with spring constitutes a linear harmonic spring pendulum



we know that,

$$T = \frac{2\pi}{\omega}$$

but $\omega^2 = \frac{k}{m}$

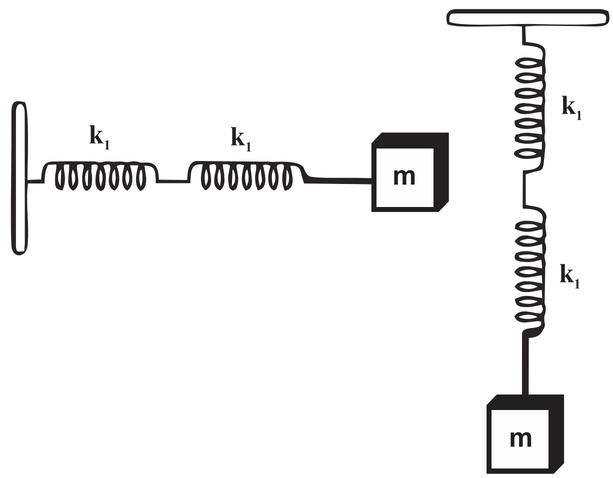
$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}}$$

and Frequency is

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

1. Time period of a spring pendulum depends on the mass suspended $T \propto \sqrt{m}$ or $n \propto \frac{1}{\sqrt{m}}$ *i.e.* greater the mass greater will be the inertia and so lesser will be the frequency of oscillation and greater will be the time period.
2. The time period depends on the force constant k of the spring *i.e.* $T \propto \frac{1}{\sqrt{k}}$ or $n \propto \sqrt{k}$
3. Time of a spring pendulum is independent of acceleration due to gravity.
4. Series combinations:



a. In series combination qual forces acts on spring but extension in springs are different

b. Spring constant of combination

$$\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\therefore \quad k_s = \frac{k_1 k_2}{k_1 + k_2}$$

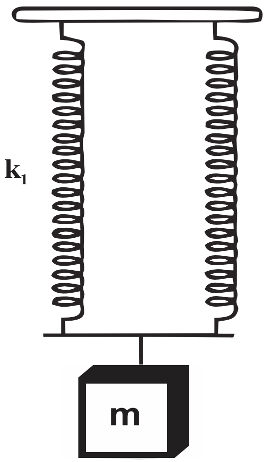
c. If n springs of different force constants are connected in series having force constant k_1, k_2, k_3,.......respectively then

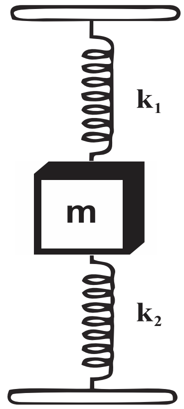
$$\frac{1}{k_S} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + ...$$

d. Time period of combination is

$$T = 2\pi\sqrt{\frac{m}{k_S}} = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

5. Parallel combination:





a. In parallel combination different forces acts on different springs but extension in springs are same

b. Spring constants of combination is $k_P = k_1 + k_2$

c. If n springs of different force constants are connected in paralle having force constant k_1, k_2,k_3,.......respectively then $k_P = k_1 + k_2 + k_3 + ...$ =nk

d. Time period of combination is

$$T_P = 2\pi\sqrt{\frac{m}{k_P}} = 2\pi\sqrt{\frac{m}{(k_1+k_2)}}$$

Key Point

1. When a spring cut into nequal pieces, spring factor of each part becomes nk

$$T = 2\pi\sqrt{\frac{m}{nk}}$$

2. When the length of spring is made n times, its spring factor becomes 1/n times and hence time period increases by $\sqrt{n}$ times.

Type - III

Numerical Problem based on spring pendulum

Formulae Used

1. $T = 2\pi\sqrt{\frac{m}{k}}$

2. For parallel combination
$k_p = k_1 + k_2$

$$T = 2\pi\sqrt{\frac{m}{k_1+k_2}}$$

3. For series combination

$$k_s = \frac{k_1 k_2}{k_1+k_2}$$

$$T_s = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1 k_2}}$$

1) The period of oscillation of a body of mass m_1 supended from a light spring is T. When a body of mass m_2 is tied to the first body and the system is made to oscillate, the period is 2T. Compare the masses m_1 and m_2.

Data: Initial mass = m_1
Initial Time period =T
Final mass = $m_1 + m_2$
Final time period = 2T

To find: $\frac{m_1}{m_2}$

Formula: $T = 2\pi\sqrt{\frac{m}{k}}$

Solution:

i. Intial time period

$$T = 2\pi\sqrt{\frac{m_1}{k}} \quad ...(1)$$

Final time period

$$2T = 2\pi\sqrt{\frac{m_2+m_1}{k}} \quad ...(2)$$

ii. Dividing equation (1) by equation (2),

$$\frac{1}{2} = \sqrt{\frac{m_1}{m_2+m_1}}$$

$$\therefore \quad \frac{m_2+m_1}{m_1} = \frac{1}{4}$$

$$\therefore \quad 4m_1 - m_1 = m_2$$

$$\therefore \quad \frac{m_1}{m_2} = \frac{1}{3}$$

Ans : The required ratio of masses is 1/3.

2) Two identical spring, each of force

constant k are connected in (a) series (b) parallel, and they support a mass m. Calculate the ratio of the time periods of the mass in the two system.

Data: $k_1 = k$; $k_2 = k$

To Find : $\frac{T_s}{T_p}$

Formulae: i. For parallel combination

$$k_p = k_1 + k_2$$

$$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

ii. For series combination

$$k_s = \frac{k_1 k_2}{k_1 + k_2}$$

$$T_s = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

Solution :

a. for series combination, the effective force constant is

$$k_s = \frac{k \times k}{k + k} = \frac{k}{2}$$

$$T_s = 2\pi\sqrt{\frac{m}{k_s}} = 2\pi\sqrt{\frac{m}{k/2}}$$

b. For parallel combination, the effective force constant is

$k_p = k + k = 2k$

$$T_p = 2\pi\sqrt{\frac{m}{k_p}} = 2\pi\sqrt{\frac{m}{2k}}$$

Required ratio of the time periods,

$$\frac{T_s}{T_p} = \sqrt{\frac{2k}{k/2}} = 2$$

Problem for Practice

1. The period of oscillation of a mass m suspended by an ideal spring is 2s. If an additional mass of 2 kg be suspended, the time period is increased by 1 s. Find the value of m.

Ans :1.6 kg

2. A 5 kg mass is attached to a spring of force constant 500 Nm^{-1}. It slides without friction on a horizontal rod. The mass is displaced from its equilibrium position by 10.0 cm aand released. Calculate (i) the period of oscillation, (ii) the maximum speed, and (iii) the maximum acceleration of the collar.

Ans: (i)0.628s (ii) 1.0ms^{-1} (iii)10 ms^{-2}

5.7 Reference Circle Method

Q.10 Show that the SHM is a projection of uniform circular motion on any diameter.

Ans:

i. Consider a particle performing uniform circular motion.

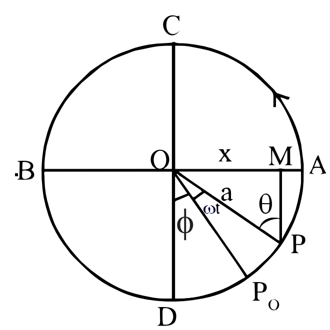
Let, ω - angular speed

A - radius of the circle,

P - position of the particle at any instant t M - projection of point P on diameter AB.

ii. As the particle performs uniform circular motion, point M moves to and fro between the two ends of diameter AB.

Let O be the origin from which the distance of the point M during motion is measured.



iii. Let, P_0 be the initial position of the particle, such that $\angle DOP_0 = \phi$

$\angle P_0OP = \omega t$ = angle traced in time t.

PM is parallel to OD taking OP as transversal

$\angle OPM = \angle POD$ ... (Alternate angles)

$\theta = \omega t + \phi$ ...(1)

iv. In Δ OPM,

$$\sin\theta = \frac{OM}{OP}$$

∴ OM = OP. sin θ

∴ $x = A\sin(\omega t + \phi)$ ...(2)

v. Differentiating w.r.t. time, we get

$\therefore \quad v = \frac{dx}{dt} = A\omega \cos(\omega t + \phi)$ ...(3)

vi. Again differentiating w.r.t. time, we get,

$\therefore \quad \frac{dv}{dt} = -A\omega^2 \sin(\omega t + \phi)$

$\therefore \quad \frac{dv}{dt} = -\omega^2 [A \sin(\omega t + \phi)]$

$\therefore$ Acceleration $= -\omega^2 x$ ...(From 2)

vii. As, Acceleration $\propto -x$

Thus, the point M moves along the diameter AB such that acceleration is directly proportional and opposite to its displacement from the mean position.

5.8 Phase in S.H.M.

Q.11 Explain the term : Phase in S.H.M.

Ans:

i. Phase in S.H.M (or for any motion) is the state of oscillation.

ii. In order to know the state of oscillation in S.H.M., it's important to know the displacement (position), the direction of velocity and the oscillation number (during which oscillation) at that instant of time.

iii. At a given position there are two posssible directions of velocity (except the extrems positions), and it repeats for successive oscillations. Hence, knowing value of displacement is not sufficient.

iv. Similarly, knowledge of only velocity will not help as there are two different positions for the same velocity (except the mean position)

v. Hence, to know the phase, a quantity is needed that is continuously changing with time. It is clear that all the quantities of liner S.H.M. (x,v a etc.) are the projections taken on a diameter, of the respective quantities for the refrence circular motion.

vi. The angular displacement $\theta = (\omega t + \phi)$ can thus be used as the phase of S.H.M. as it varies continuously with time. In this case, it will be called as the phase angle.

INTEX QUESTION

Describe the state of oscillation if the phase angle is 1110°.

Ans:

i. Angle 1110° can be written as,
$1110° = (3 \times 360°) + 30°$

ii. Angle 360° represent one oscillation

$\therefore$ Angle $(3 \times 360°)$ means 3 oscillations are completed and particle is performing 4th oscillation.

iii. $\therefore \quad x = A \sin 30° = \frac{A}{2}$

iv. $\therefore$ Phase angle 1110° indicates that during its 4th oscillation, the particle is at $+\frac{A}{2}$ and moving to the positive extreme.

INTEX QUESTION

While completing its third oscillation during linear S.H.M., a particle is at $\frac{-\sqrt{3}A}{2}$, heading to the mean position. Determine the phase angle.

Ans:

i. $x = \frac{-\sqrt{3}A}{2}$

As $x = A \sin \theta_1$

$\therefore \quad A \sin \theta_1 = \frac{-\sqrt{3}A}{2}$

$\theta_1 = 300°$

$\therefore \quad \theta_1 = \left(2\pi - \frac{\pi}{3}\right)^c$

ii. From negative side, the particle is heading to the mean position. Thus, the phase angle is in the fourth quardant for that oscillation.

iii. As particle is completing third oscillation

$\therefore$ Phase angle, $\theta = 2 \times 2\pi + \theta_1$

$\therefore \quad \theta = 4\pi + \left(2\pi - \frac{\pi}{3}\right) = 6\pi - \frac{\pi}{3} = \left(\frac{17\pi}{3}\right)^c$

$\therefore$ The phase angle is $\left(\frac{17\pi}{3}\right)^c$

5.9 Graphical Representation of S.H.M.

Q.12 Explain with graphs the variation of displacement, velocity and acceleration with time, when the particle starts from the mean position.

Ans: When the particle starts from the mean position:

i. The general expression for the displacement in SHM is,

$x = A \sin(\omega t + \phi)$ ...(1)

ii. As the particle starts from the mean position.

$\therefore$ $\phi = 0$

$\therefore$ Equation (i) becomes,

$x = A \sin \omega t$ ...(2)

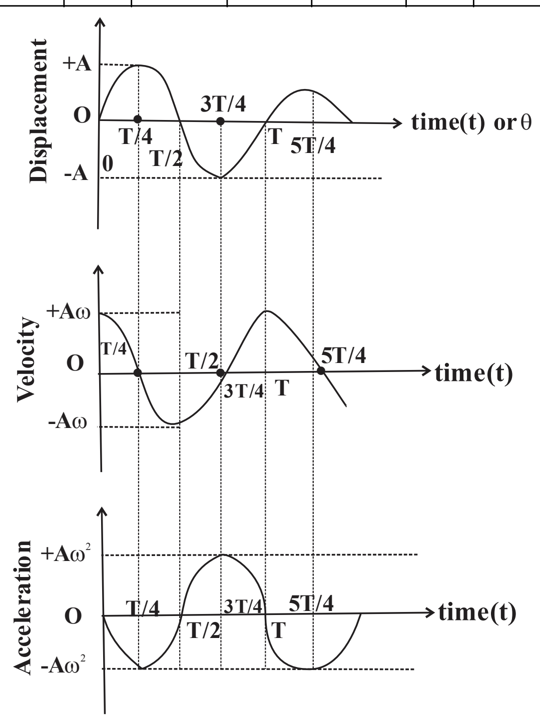
iii. But, velocity is the rate of change of displacement with time

$\therefore$ $v = \frac{dx}{dt} = A\omega.\cos\omega t$ ...(3)

iv. Acceleration is the rate of change of velocity with time

$\therefore$ Acceleration $= \frac{dv}{dt} = -A\omega^2 \sin\omega t$ ...(4)

Time	0	T / 4	T / 2	3T / 4	T	5T / 4
ωt	0	$\pi / 2$	π	$3\pi / 2$	2π	$5\pi / 2$
x	0	A	0	−A	0	A
v	$A\omega$	0	$-A\omega$	0	$A\omega$	0
acc^n	0	$-A\omega^2$	0	$A\omega^2$	0	$-A\omega^2$



Conclusions:

i. From graphs we can say displacement, velocity and acceleration of S.H.M. are periodic function of time.

ii. Displacement-time curve and acceleration time curve are sine curve and velocity - time curve is a cosine curve.

iii. There is phase difference of $\pi / 2$ radian between displacement and velocity.

iv. There is phase difference of $\pi / 2$ radian between velocity and acceleration.

v. There is phase difference of π radian between displacement and acceleration.

vi. All curves represent same path after phase 2π radian.

Q.13 Explain with graph the variation of displacement, velocity and acceleration When the particle starts from the extreme position.

Ans: When the particle starts from the extreme position:

i. The general expression for the displacement in SHM is,

$x = A \sin(\omega t + \phi)$ ...(1)

ii. As the particle starts from the extreme position

$\therefore$ $\phi = \frac{\pi}{2}$

$\therefore$ equation (i) becomes,

$x = A \sin\left(\omega t + \frac{\pi}{2}\right)$

$\therefore$ $x = A \cos \omega t$ ...(2)

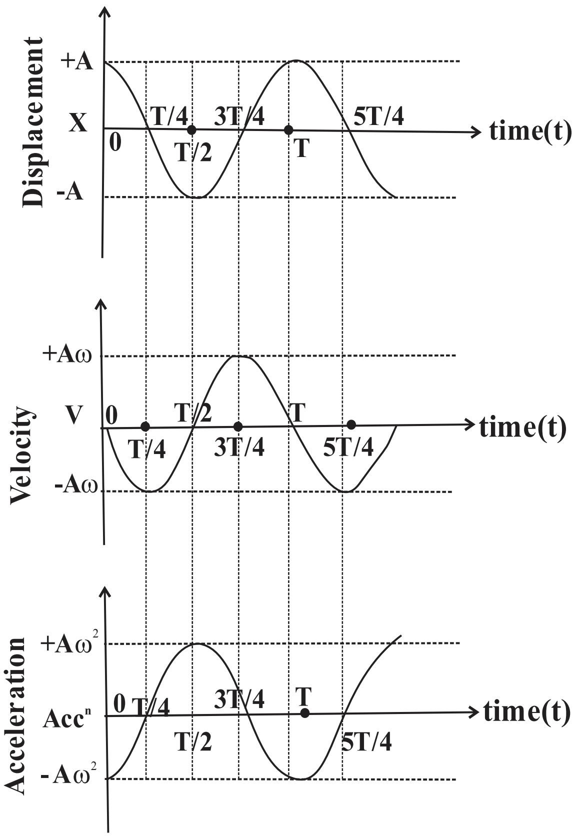
iii. But, velocity is the rate of change of displacement with time

$\therefore$ $v = \frac{dx}{dt} = -A\omega.\sin\omega t$ ...(3)

iv. Acceleration is the rate of change of velocity with time

$\therefore$ Acceleration $= \frac{dv}{dt} = -A\omega^2 \cos\omega t$ ...(4)

Time	0	T / 4	T / 2	3T / 4	T	5T / 4
ωt	0	$\pi / 2$	π	$3\pi / 2$	2π	$5\pi / 2$
x	A	0	−A	0	A	0
v	0	$-A\omega$	0	$A\omega$	0	$-A\omega$
acc^n	$-A\omega^2$	0	$A\omega^2$	0	$-A\omega^2$	0



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iv. There is phase difference of $\pi / 2$ radian between velocity and acceleration.

v. There is phase difference of π radian between displacement and acceleration.

vi. All curves represent same path after phase 2π radian.

vii. The displacement and acceleration is maximum at extreme position where as velocity is minimum at the same position.

5.10 Composition of two S.H.M.s having same period and along the same path.

Q.14 Discuss analytically the composition of two SHMs parallel to each other, and having the same period. Hence, obtain the expressions for resultant amplitude when the phase difference is,

i. zero, ii. $\frac{\pi}{2}$ and iii. π.

Ans:

i. Two SHMs having same period $\frac{2\pi}{\omega}$ and parallel to each other can be represented by,

$x_1 = A_1 \sin(\omega t + \phi_1)$ ...(1) and

$x_2 = A_2 \sin(\omega t + \phi_2)$ ...(2)

where, A_1, A_2 - are amplitude and

ϕ_1, ϕ_2 - are initial phases.

ii. The resultant displacement at any instant t is given by,

$x = x_1 + x_2$

$\therefore$ $x = A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_2)$

$\therefore$ $x = A_1 [\sin\omega t . \cos\phi_1 + \cos\omega t . \sin\phi_1]$

$+ A_2 [\sin\omega t . \cos\phi_2 + \cos\omega t . \sin\phi_2]$

$\therefore$ $x = \sin\omega t [A_1 . \cos\phi_1 + A_2 . \cos\phi_2]$

$+ \cos\omega t [A_1 . \sin\phi_1 + A_2 . \sin\phi_2]$

iii. Substitute,

$[A_1 . \cos\phi_1 + A_2 . \cos\phi_2] = R . \cos\delta$ ...(3)

and $[A_1 . \sin\phi_1 + A_2 . \sin\phi_2] = R . \sin\delta$ ...(4)

iv. Resultant displacement x will be,

$x = \sin\omega t \times R . \cos\delta + \cos\omega t \times R . \sin\delta$

$\therefore$ $x = R [\sin\omega t . \cos\delta + \cos\omega t . \sin\delta]$

$\therefore$ $x = R [\sin(\omega t + \delta)]$

This shows that resultant motion is also SHM with same period $\frac{2\pi}{\omega}$.

Where, R - resultant amplitude and

δ - resultant initial phase.

Expression for Resultant amplitude (R):

Squaring and adding equations (iii) and (iv),

$(A_1 \cos\phi_1 + A_2 \cos\phi_2)^2 + A_1 \sin\phi_1 + A_2 \sin\phi_2)^2$

$= R^2 \cos^2\delta + R^2 \sin^2\delta$

$\therefore$ $[A_1^2 \cos^2\phi_1 + 2A_1 . A_2 . \cos\phi_1 . \cos\phi_2 + A_2^2 \cos^2\phi_2$

$+ A_1^2 \sin^2\phi_1 + 2A_1 . A_2 . \sin\phi_1 . \sin\phi_2 + A_2^2 \sin^2\phi_2]$

$= R^2 (\cos^2\delta + \sin^2\delta)$

$\therefore \quad R^2 = A_1^2(\cos^2\phi_1 + \sin^2\phi_1) + 2A_1.\ A_2(\cos\phi_1.\cos\phi_2 + \sin\phi_1.\sin\phi_2) + A_2^2(\cos^2A_2 + \sin^2 A_2)$

$\therefore \quad R^2 = A_1^2 + 2A_1.A_2.\cos(\phi_1 - \phi_2) + A_2^2$

$\therefore \quad R = \sqrt{A_1^2 + A_2^2 + 2A_1.A_2.\cos(\phi_1 - \phi_2)}$ ...(5)

Expression for initial phase (δ) :

Dividing equation (4) by (3),

$$\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2} = \frac{R.\sin\delta}{R.\cos\delta}$$

$$\therefore \quad \tan\delta = \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$$

$$\therefore \quad \delta = \tan^{-1}\left[\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}\right]$$

Resultant amplitude for different cases of phase difference :

a. When Particles arc in some phase

$\therefore \quad (\phi_1 - \phi_2) = 0$

$\therefore \quad \cos(\phi_1 - \phi_2) = 1$

$\therefore \quad R = \sqrt{A_1^2 + A_2^2 + 2A_1.A_2(1)}$

$= \sqrt{(A_1 + A_2)^2} = A_1 + A_2$

$\therefore \quad R = A_1 + A_2$

b. When particle difference phase by 90°

$\therefore \quad (\phi_1 - \phi_2) = 90^0$

$\therefore \quad \cos(\phi_1 - \phi_2) = 0$

$\therefore \quad R = \sqrt{A_1^2 + A_2^2 + 2A_1.A_2(0)}$

$= \sqrt{A_1^2 + A_2^2}$

c. When particles arc in opposite phase

$(\phi_1 - \phi_2) = 180^0$

$\therefore \quad \cos(\phi_1 - \phi_2) = -1$

$\therefore \quad R = \sqrt{A_1^2 + A_2^2 + 2A_1.A_2(-1)}$

$= \sqrt{A_1^2 + A_2^2 + (-2A_1.A_2)}$

$= \sqrt{A_1^2 + A_2^2 - 2A_1.A_2}$

$= \sqrt{(A_1 - A_2)^2} = |A_1 - A_2|$

$R = |A_1 - A_2|$

Type - IV
Numerical based on Composition of two S.H.M.

Formulae used

1. $R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_1)}$
2. When particles are in same phase

   $R = A_1 + A_2$

   When particles are in opposite phase

   $R = A_1 - A_2$

   When particles differ in phase by 90°

   $R = \sqrt{A_1^2 + A_2^2}$
3. $\delta = \tan^{-1}\left[\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}\right]$

1) Two parallel S.H.M.s represented by

$$\mathbf{x_1 = 5\sin\left(4\pi t + \frac{\pi}{3}\right)cm} \quad \textbf{and}$$

$$\mathbf{x_2 = 3\sin\left(4\pi t + \frac{\pi}{4}\right)cm}$$

are superposed on a particle. Determine the amplitude and epoch of the resultant S.H.M.

Data: $x_1 = 5\sin\left(4\pi t + \frac{\pi}{3}\right)cm$

$x_2 = 3\sin\left(4\pi t + \frac{\pi}{4}\right)cm$

To find: i. R ii. δ

Formulae: i. $R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}$

ii. $\delta = \tan-^1 \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$

Solution:

i. $x_1 = 5\sin\left(4\pi t + \frac{\pi}{3}\right)$

Comparing with

$x_1 = A_1\sin(\omega t + \phi_1)$

$A_1 = 5cm$ and $\phi = \frac{\pi}{3}$

ii. $x_2 = 3\sin\left(4\pi t + \frac{\pi}{4}\right)$

Comparing with

$x_2 = A_2\sin(\omega t + \phi_2)$

$A_2 = 3\text{cm}$ and $\phi = \frac{\pi}{4}^c$

iii. $R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}$

$R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}$

$R = \sqrt{(5)^2 + (3)^2 + 2\times5\times3\times\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}$

$= \sqrt{25 + 9 + 30\times\cos\frac{\pi}{12}}$

$= \sqrt{25 + 9 + (30\times0.9659)}$

$= \sqrt{34 + 28.977}$

$= \sqrt{62.977} = 7.936$ cm

iv. $\delta = \tan^{-1}\left[\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}\right]$

$\delta = \tan^{-1}\left[\frac{5\sin\frac{\pi}{3} + 3\sin\frac{\pi}{4}}{5\cos\frac{\pi}{3} + 3\cos\frac{\pi}{4}}\right]$

$= \tan^{-1}\left[\frac{5\times\frac{\sqrt{3}}{2} + 3\times\frac{1}{\sqrt{2}}}{5\times\frac{1}{2} + 3\times\frac{1}{\sqrt{2}}}\right]$

$= \tan^{-1}\left[\frac{5\times0.866 + 3\times0.707}{5\times0.5 + 3\times0.707}\right]$

$= \tan^{-1}\left[\frac{4.33 + 2.121}{2.5 + 2.121}\right]$

$= \tan^{-1}\left[\frac{6.451}{4.621}\right] = \tan^{-1}[1.396] = 54°23'$

Ans :	The amplitude and epoch of the resultant S.H.M. is 7.936 cm and 54°23' respectively

.

9. The displacement of an oscillating particle is given by $x = a\sin\omega t + b\cos\omega t$ where a,b and ω are constants. Prove that the particle performs a linear S.H.M. with amplitude $A = \sqrt{a^2 + b^2}$

Data: $x = a\sin\omega t + b\cos\omega t$

To prove : Equation represent S.H.M.

Proof : $x = a\sin\omega t + b\cos\omega t$ ...(1)

Differentiating w.r.t 't'

$\frac{dx}{dt} = a\cos\omega t\times\omega + b(-\sin\omega t)\times\omega$

$v = a\omega\cos\omega t - b\omega\sin\omega t$ ...(2)

Differentiating w.r.t 't'

$\frac{dv}{dt} = a\omega(-\sin\omega t)\times\omega - b\omega\cos\omega t\times\omega$

Accleration $= -a\omega^2\sin\omega t - b\omega^2\cos\omega t$

acceleration $= -\omega^2(a\sin\omega t + b\cos\omega t)$

acceleration $= -\omega^2 x$ ...(from1)

∴ acceleration $\alpha - x$

∴ The given equation represent S.H.M.

ii. $x = a\sin\omega t + b\cos\omega t$

∴ $x = a\sin(\omega t + 0) + b\sin\left(\omega t + \frac{\pi}{2}\right)$

Comparing with

$x = A_1\sin(\omega t + \phi_1) + A_2\sin(\omega t + \phi_2)$

$A_1 = a;\ A_2 = b;\ \phi_1 = 0;\quad \phi_2 = \frac{\pi}{2}$

$R = \sqrt{A_1^2 + A_2^2}$

∴ $R = \sqrt{a^2 + b^2}$

Hence proved

Problem for Practice

1. The equation of a simple harmonic motion is given by $y = 6\sin10\pi t + 8\cos10\pi t$, where y is in cm and t in sec. Determine the amplitude, period and initial phase

Ans:53°8'

2. S.H.M is given by the equation $x = 8\sin(4\pi t) + 6\cos(4\pi t)$ cm. Find its (a) amplitude (b) initial phase (c) period (d) frequency

Ans:10cm, 360°53', 0.5 sec, 2Hz

5.11 Energy of a Particle Performing S.H.M.

Q.15 Show that total energy of a particle performing SHM is constant.

Ans: A particle performing SHM has both potential energy and kinetic energy.

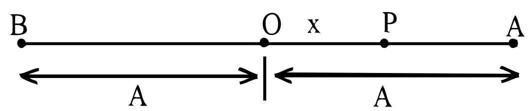
∴ Total energy of a particle in SHM is,

T.E. = P.E. + K.E.

Expression for potential energy in SHM :

i. Consider a particle of mass m performing SHM with amplitude A.

Let, the particle be at distance x from the mean position at any instant t.



ii. Force acting on the particle is,

$F = -kx$ ...(1)

iii. Work done for further infinitesimal displacement dx will be,

$dW = -F.\,dx$

$= -(-kx).\,dx$

$= kx\,dx$ ...(2)

iv. Total work done in displacement from zero to *x* will be

$$W = \int dW = \int_0^x kx\,dx = \left[\frac{kx^2}{2}\right]_0^x = \frac{1}{2}kx^2 \quad ...(3)$$

v. This work done is stored in the form of potential energy.

∴ $P.E = \frac{1}{2}kx^2$

∴ $\boxed{P.E = \frac{1}{2}m\omega^2x^2}$ $\left[\text{As } \frac{k}{m} = \omega^2\right]$ ...(4)

Case-1: At mean position

$x = 0$

$\boxed{P.E = 0}$

Case-2: At extreme position

x = IA

$\boxed{(P.E)_{max} = \frac{1}{2}m\omega^2A^2}$

Expression for kinetic energy in SHM:

i. Consider a particle of mass m performing SHM.

Let, the particle be at a distance x from mean position.

Then, the velocity of the particle is given by,

$v = \pm\omega\sqrt{A^2 - x^2}$,

∴ $v^2 = \omega^2(A^2 - x^2)$ ...(5)

Kinetic energy = $\frac{1}{2}mv^2$

∴ $K.E. = \frac{1}{2}m\,\omega^2(A^2 - x^2)$ ...(6)

Case-1: At mean position

$x = 0$

$\boxed{(K.E)_{max} = \frac{1}{2}m\omega^2A^2}$

Case-2: At extreme position

$x = \pm A$

∴ $\boxed{(K.E) = 0}$

Expression for total energy in SHM:

T.E. = P.E. + K.E.

$= \frac{1}{2}m\,\omega^2x^2 + \frac{1}{2}m\omega^2(A^2 - x^2)$

$= \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2A^2 - \frac{1}{2}m\omega^2x^2$

∴ $T.E. = \frac{1}{2}m\,\omega^2A^2 = \frac{1}{2}kA^2$ ...(7)

This is the expression for the total energy in SHM.

Note :

Dependence of T.E. on different factors:

$T.\,E. = \frac{1}{2}KA^2 = \frac{1}{2}m\omega^2A^2$ *(As,* $\omega = 2\pi n$*)*

$= \frac{1}{2}m\,(2\pi n)^2A^2$

$= \frac{1}{2}m\;4\;\pi^2n^2A^2$

∴ $T.\,E. = 2\pi^2\,mn^2\,A^2$

i. *As,* $2\pi^2mn^2$ = *constant*

∴ $T.\,E. \propto A^2$,

ii. *As,* $2\pi^2ma^2$ = *constant,*

$T.\,E. \propto n^2$ *Also* $T.E \propto \frac{1}{T^2}$

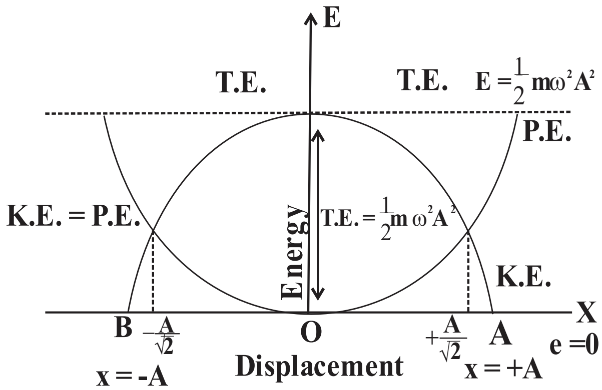
Q.16 Represent graphically K.E., P.E. and T.E. for a particle performing linear S.H.M. with respect to displacement in S.H.M.

Ans: As shown in the graph.

i. At the mean position, the energy is entirely kinetic.

ii. At the extreme position, the energy is entirely potential.

iii. As the particle moves away from the mean position, the K.E. is gradually converted into P.E.



Key Point

Type - V
Numerical based

Formulae Used

1. $E_p = \frac{1}{2} m\omega^2 x^2$

At mean position

$E_p = 0$

At Extreme position

$(E_p)_{max} = \frac{1}{2} m\omega^2 A^2$

2. $E_k = \frac{1}{2} m\omega^2 (A^2 - x^2)$

At mean position

$(E_k)_{max} = \frac{1}{2} m\omega^2 A^2$

At extreme position

$E_k = 0$

3. $E_T = \frac{1}{2} m\omega^2 A^2$

★ **1) The total energy of a particle of mass 200 g, performing S.H.M. is 10^{-2} J. Find its maximum velocity and period if the amplitude is 7 cm.**

Data: m = 200 g = 0.2 kg, $E_T = 10^{-2}$J,
A = 7 cm = 0.07 m

To Find: i. v_{max} ii. T

Formulae: i. $E_T = \frac{1}{2} m\omega^2 A^2$ ii. $\omega = \frac{2\pi}{T}$

iii. $v_{max} = \omega A$

Solution:

i. $E_T = \frac{1}{2} m\omega^2 A^2 = \frac{1}{2} m(\omega A)^2$

$E_T = \frac{1}{2} m v_{max}^2$ ....[$\because v_{max} = \omega A$]

$\therefore$ $v_{max} = \sqrt{\frac{2(E_T)}{m}} = \sqrt{\frac{2 \times 10^{-2}}{0.2}}$

$= \sqrt{0.1} = 0.3162 m/s$

ii. $v_{max} = \omega A = \left(\frac{2\pi}{T}\right) A$ ...$\left(\because \omega = \frac{2\pi}{T}\right)$

$\therefore$ $T = \frac{2\pi A}{v_{max}}$

$T = \frac{2\pi \times 0.07}{0.3162}$

$= \frac{2 \times 3.14 \times 7 \times 10^{-2}}{3.162 \times 10^{-1}}$

= At log { table } $\times 10^{-1}$

log N		log D	
log 2	0.3010	log 3.162	0.5000
log 3.14	0.4969		
log 7	0.8451		
	1.6430		

$= \text{At}\log \begin{bmatrix} 1.6430 \\ -\ 0.5000 \\ 1.1430 \end{bmatrix} \times 10^{-1} = 13.90 \times 10^{-1}$

=1.39 sec

Ans : i. Maximum velocity of S.H.M. is 0.3162s
ii. Time period of S.H.M is 1.39 s.

★ **2) Potential energy of a particle performing linear S.H.M is $0.1\pi^2 x^2$ joule. If mass of the particle is 20 g, find the frequency of S.H.M.**

Data: $E_p = 0.1\pi^2 x^2$ joule, m = 20 g = 0.02kg

To find: n

Formula: $E_p = \frac{1}{2}m\omega^2 x^2$

$\omega = 2\pi n$

Solution: $E_p = \frac{1}{2}m\omega^2 x^2$

$E_p = \frac{1}{2}m(2\pi n)^2 x^2$

$\therefore \quad E_p = \frac{1}{\cancel{2}_1} m \times \cancel{4}^2 \pi^2 n^2 x^2$

$n^2 = \frac{E_p}{2\pi^2 m x^2}$

$n^2 = \frac{0.1 \cancel{\pi^2 m x^2}}{2 \cancel{\pi^2 m x^2}}$

$n^2 = \frac{0.1}{2m} = n^2 = \frac{0.1}{2 \times 0.02} = \frac{10}{4}$

$n = \frac{\sqrt{10}}{2}$

$= \frac{3.162}{2} = 1.581 \text{ Hz}$

Ans :	The frequency of the SHM is 1.581 Hz

★ 3) **The total energy of a body of mass 2 kg performing S.H.M is 40 J. Find its speed while crossing the centre of the path.**

Data: m = 2kg, $E_T = 40$ J

To find: (v_{max})

Formula: i. $E_T = \frac{1}{2}m\omega^2 A^2$ ii. $\omega = \frac{2\pi}{T}$

iii. $v_{max} = \omega A$

Solution:

$v_{max} = \sqrt{\frac{2 \times E_T}{m}} = \sqrt{\frac{2 \times 40}{2}}$

$= 2\sqrt{10} = 2 \times 3.162 = 6.324$ m/s.

Ans :	Speed of particle while crossing the mean postion is 6.324m/s.

★ 4) **At what distance from the mean position is the kinetic energy of a particle performing S.H.M. of amplitude 8 cm, three times its potential energy?**

Data: A = 8 cm, $E_K = 3\ E_P$

To Find: x

Formulae: i. $E_K = \frac{1}{2}m\omega^2(A^2 - x^2)$

ii. $E_P = \frac{1}{2}m\omega^2 x^2$

Solution: Given $E_K = 3E_P$

$\frac{1}{2}m\omega^2(A^2 - x^2) = 3 \times \frac{1}{2}m\omega^2 x^2$

$\therefore \quad 4x^2 = A^2$

$\therefore \quad x = \frac{A}{2} = \frac{8\text{cm}}{2} = 4\text{cm}$

Ans :	Distance from the mean position where the kinetic energy is thrice of potential energy is 4 cm.

★ 5) **A wooden block of mass m is kept on a piston that can perform vertical vibrations of adjustable frequency and amplitude. During vibrations, we don't want the block to leave the contact with the piston. How much maximum frequency is possible if the amplitude of vibrations is resticted to 25 cm?In this case, how much is the energy per unit mass of the block? ($g \approx \pi^2 \approx 10\ m\ s^{-2}$)**

Data: A = 25cm = 0.25 m

To Find : i. n ii. E/m

Formula: i. $F_{max} = -kA$

$F_{max} = m\omega^2 A$

ii. $E = \frac{1}{2}m\omega^2 A^2$

iii. $\omega = 2\pi n$

Solution:

i. At height position

$\therefore \quad mg = m\omega^2 A$

$\therefore \quad \omega = \sqrt{\frac{g}{A}} = \sqrt{\frac{10}{25 \times 10^{-2}}} = 2\sqrt{10}\text{rad / s}$

ii. $\omega = 2\pi n$

$$\therefore \quad n = \frac{\omega}{2\pi} = \frac{\cancel{2}\sqrt{10}}{\cancel{2}\pi} = \frac{3.162}{3.142}$$

$$= \text{At}\log\begin{bmatrix} 0.5000 \\ -0.4972 \\ \hline 0.0028 \end{bmatrix}$$

$$= 1.007 \text{ Hz}$$

iii. $\text{Energy of SHM} = \frac{1}{2}m\omega^2A^2$

$$\therefore \quad \frac{E}{m} = \frac{1}{2}\omega^2A^2 = \frac{1}{2}(2\sqrt{10})^2(25\times10^{-2})^2$$

$$= \frac{1}{2}\times4\times10\times625\times10^{-4}$$

$$= 12500\times10^{-4} = 1.25 \text{ J/kg}$$

Ans : i. The maximum frequency possible for this oscillation is 1 Hz
ii. The energy per unit mass of the block is 1.25 J/kg.

Problem for Practice

1. A particle is executing SHM of amplitude A. At what displacement from the mean position, is the energy half kinetic and half potential?

Ans : $\mathbf{\frac{A}{\sqrt{2}}}$

2. A spring of force constant 800 Nm^{-1} has an extension of 5 cm. What is the work done in increasing the extension from 5 cm to 15 cm?

Ans : 8J

3. A particle executes SHM of period 8 seconds. After what time of its passing through the mean position will the energy be half kinetic and half potential?

Ans : 1 s

4. A particle of mass 10 gm executes linear S.H.M. of amplitude 2 cm. With a period of 2 second. Find its potential energy. Kinetic energy $\left(\frac{1}{6}\right)^{th}$ second after it has passed through the mean position

Ans : E_p = 4.935×10^{-6}J, E_k = 1.48×10^{-5}J

5.12 Simple Pendulum

Q.17 Explain an ideal simple pendulum and the practical simple pendulum.

Ans: An ideal simple pendulum :

It is defined as a heavy particle suspended by a weightless, inextensible string from a rigid support.

Practical simple pendulum :

It is defined as a small heavy sphere (called bob) suspended by a light and inextensible string from a rigid support.

An ideal simple pendulum cannot be realized in practice because we cannot get a heavy particle or a weightless string. Therefore, in the place of a heavy particle, a small heavy metallic sphere (called bob) is used and in the place of a weightless string, a light string is used.

Q.18 Show that motion of a simple pendulum is linear SHM. Obtain its period.

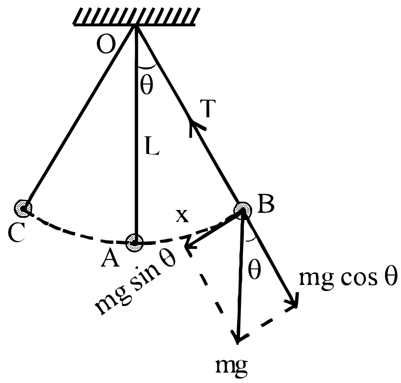
Ans:

i. Consider a simple pendulum,
Let, m - mass of the bob.
L - length of the pendulum.

ii. If the bob is in displaced position with very small angle θ, then the displacement x of the bob may be treated in a straight line.

iii. The forces acting on the bob are,
a. Weight of the bob(mg) in downward direction.
b. Tension (T) along the string

Resolve mg in two components.
1. mg cosθ along the string which balances the tension in the string.
2. mg sin θ, perpendicular to the string. This unbalanced force tries to bring the bob to its original position. Hence, it is called the restoring force, say F.



iv. Restoring force is,

$F = -mg\sin\theta$

The negative sign implies that force opposes the increase in θ.

Now, θ is very small,

∴ $\sin\theta \approx \theta$

∴ $F = -mg\theta$ ...(1)

v. angle in radtion = $\frac{\text{arc}}{\text{radius}}$

∴ $\theta = \frac{x}{L}$

Substituing in above eq (1)

∴ $F = \frac{-mgx}{L}$ ...(2)

∴ $F = -\left(\frac{mg}{L}\right)x$

∴ $F = -k\,x$

Where m, g, L are constant

∴ $F \alpha - x$

vi. Thus for small displacement the restoring force is directly proportional to the displacement and is opposite direction

Therefore, motion of the bob of a simple pendulum is linear SHM.

vii. **Expression for period of a simple pendulum:**

Now,

$ma = -\frac{mg}{L}$ ...[From (1)]

We have, $\text{acceleration} = \frac{-g}{L}.x$

∴ $|\text{Acceleration per unit displacement}| = \frac{g}{L}$

Now, $T = \frac{2\pi}{\omega}$

$T = \frac{2\pi}{\sqrt{\text{acceleration / unit displacement}}}$

∴ $T = \frac{2\pi}{\sqrt{(g/L)}}$

∴ $\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$

This is the expression for period of a simple pendulum.

Q.19 State laws of simple pendulum.

Ans:

i. Period of simple pendulum is directly proportional to the square root of its length.

i.e. $T \propto \sqrt{L}$

ii. Period of simple pendulum is inversely proportional to the square root acceleration due to gravity.

i.e. $T \propto \frac{1}{\sqrt{g}}$

iii. Period of simple pendulum is independent of the mass of the bob.

iv. Period of simple pendulum is independent of the amplitude of oscillation, provided that the amplitude is small.

Q.20 What is second's pendulum ? Obtain expression for its length.

Ans:

Second's pendulum :

A simple pendulum whose period is 2 seconds is called second's pendulum.

Expression for length of seconds pendulum:

The period of simple pendulum is,

$T = 2\pi\sqrt{\frac{L}{g}}$

∴ $T^2 = 4\pi^2\frac{L}{g}$

∴ $L = \frac{g.T^2}{4\pi^2}$

For second's pendulum, T = 2 seconds.

∴ $L = \frac{g(2)^2}{4\pi^2}$

∴ $\boxed{L = \frac{g}{\pi^2}}$

This is the expression for the length of a second's pendulum.

Type - VI
Numerical based on Simple pendulum

Formulae Used

1. $T = 2\pi\sqrt{\frac{L}{g}}$

2. Length of second's pendulum $L = \frac{g}{\pi^2}$

★ 1) The period of oscillations of a simple pendulum increases by 10 %, when its length is increased by 21 cm. Find its initial length and initial period.

Data: $T_2 = T_1 + 10\%\ T_1 = T_1 + \frac{10}{100}T_1$

$$T_2 = \left(1+\frac{10}{100}\right)T_1$$

$\therefore \quad \frac{T_2}{T_1} = \frac{110}{100}$, $L_2 = (L_1+0.21)$m

To Find: i. L_1 ii. T_1

Formula: $T = 2\pi\sqrt{\frac{L}{g}}$

Solution:

i. $$T_1 = 2\pi\sqrt{\frac{L_1}{g}} \quad ...(1)$$

$$T_2 = 2\pi\sqrt{\frac{L_2}{g}} \quad ...(2)$$

Dividing equation (1) by equation (2),

$$\therefore \quad \frac{T_1}{T_2} = \sqrt{\frac{L_1}{L_2}}$$

$$\therefore \quad \frac{100}{110} = \sqrt{\frac{L_1}{L_2}} \quad ....\text{(Given)}$$

$$\therefore \quad \frac{10}{11} = \sqrt{\frac{L_1}{L_1+0.21}}$$

$$\frac{100}{121} = \frac{L_1}{L_1+0.21}$$

$$100\,L_1 + 21 = 121\,L_1$$

$$21L_1 = 21$$

$$\therefore \quad L_1 = 1\text{m}$$

ii. $$T_1 = 2\pi\sqrt{\frac{L_1}{g}}$$

$$\therefore \quad T_1 = 2\pi\sqrt{\frac{1}{9.8}}$$

$$= 2\times\pi\times\sqrt{\frac{10}{98}} = \frac{2\pi}{7}\sqrt{\frac{10}{2}}$$

$$= \frac{2\times 3.142\times 2.236}{7} = 2.007 \text{ s}$$

Ans : i. Initial length of pendulum is1 m
ii.Initial time period is 2.007 s.

★ 2) In summer season, a pendulum clock is regulated as a second's pendulum and it keeps correct time. During winter, the length of the pendulum decreases by 1 %. How much time will the clock gain or lose in one day.
(g = 9.8 m/s²)

Data: $T_s = 2$ s,

$L_w = L_s - 1\ \%\ L_s$

$$L_w = \left(1-\frac{1}{100}\right)L_s = \frac{99}{100}L_s$$

To Find : Time loss or gain by clock in one day

Formula: $T = 2\pi\sqrt{\frac{L}{g}}$

Solution:

i. In a day of 86400 seconds, to keep correct time, the clock's pendulum should perform,

$$\frac{86400}{2} = 43200 \text{ Oscillations.}$$

ii. In summer, time period of pendulum is

$$T_s = 2\pi\sqrt{\frac{L_s}{g}} \quad ...(1)$$

In winter time period of pendulum is

$$T_w = 2\pi\sqrt{\frac{L_w}{g}} \quad ...(2)$$

Dividing equation (2) by equation (1),

$\therefore \quad \dfrac{T_w}{T_s} = \sqrt{\dfrac{L_w}{L_s}} = \sqrt{\dfrac{99}{100}}$

$\therefore \quad \dfrac{T_w}{2} = \sqrt{0.99}$

$\therefore \quad T_w = 1.99$ s

iii. With this period, the pendulum will now perform

$\dfrac{86400}{1.99} = 43417$ oscillations per day.

iv. Thus, it will gain

$43417 - 43200 = 217$ oscillations per day.

v. As per oscillations the clock refers to 2 second.

Thus, the time gained per day

= 217 × 2 = 434 second

= 7 minutes 14 seconds

Ans :	Time gained by clock is 7 min 14 sec

★ 3) A simple pendulum performs S.H.M of period 4 seconds. How much time after crossing the mean position, will the displacement of the bob be one third of its amplitude?

Data: $T = 4$ s, $x = \dfrac{A}{3}$

To find: (t)

Formulae: i. $x = A\sin(\omega t)$ ii. $\omega = \dfrac{2\pi}{T}$

Solution:

i. $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$ rad / s

ii. $x = A\sin(\omega t)$

$\dfrac{A}{3} = A\sin\left(\dfrac{\pi}{2}t\right)$

$\therefore \quad \sin\left(\dfrac{\pi}{2}\right)t = \dfrac{1}{3}$

$\dfrac{\pi}{2}t = \sin^{-1}(0.3333)$

$\dfrac{\pi}{2}t = 19°28'$

$\dfrac{\pi}{2}t = 19°\left(\dfrac{28}{60}\right)^{\circ}$

$\dfrac{\pi}{2}t = 19.47° \times \dfrac{\pi}{180}$

$t = \dfrac{38.94}{180} = \dfrac{38.94}{1.8\times 10^2}$

$= A\log\left\{\begin{matrix}1.5904\\ -0.2553\\ \hline 1.3351\end{matrix}\right\}\times 10^{-2}$

$= 21.63 \times 10^{-2}$ sec

$= 0.2163$ sec

Ans :	Time taken by particle is 0.2163 s

★ 4) A simple pendulum of length 100 cm perform S.H.M. Find the restoring force acting on its bob of mass 50 g when the displacement from the mean position is 3cm.

Data: $L = 100\text{cm} = 1\text{m}$,
$m = 50\text{g} = 5\times 10^{-2}\text{kg}$,
$x = 3\text{ cm} = 3\times 10^{-2}\text{m}$

To find: F

Formulae: i. $T = 2\pi\sqrt{\dfrac{L}{g}}$ ii. $\omega = \dfrac{2\pi}{T}$
iii. $k = m\omega^2$ iv. $|F| = kx$

Solution:

i. $T = 2\pi\sqrt{\dfrac{L}{g}}$

$\therefore \quad \dfrac{2\pi}{T} = \sqrt{\dfrac{g}{L}}$

$\therefore \quad \omega = \sqrt{\dfrac{g}{L}} = \sqrt{\dfrac{9.8}{1}} = \sqrt{9.8}$ rad / s

ii. $F = kx$

$= m\omega^2 x$

$= 5\times 10^{-2} \times \left(\sqrt{9.8}\right)^2 \times 3\times 10^{-2}$

$= 15 \times 9.8 \times 10^{-4}$

$= 1.48 \times 10^{-2}$ N

Ans :	The magnitude of force acting on the bob is 1.48×10^{-2}N.

★ 5) **Find the change in length of a second's pendulum, if the acceleration due to gravity at the place changes from 9.75 m/s² to 9.8 m/s². (Take $\pi^2 = 10$)**

Data: $g_1 = 9.75\ m/s^2$, $g_2 = 9.8\ m/s^2$

To find: Change in length of second's pendulum $(L_2 - L_1)$

Formula: $L = \frac{g}{\pi^2}$

Solution: $L_1 = \frac{g_1}{\pi^2}$ ...(1)

$L_2 = \frac{g_2}{\pi^2}$ ...(2)

$$L_2 - L_1 = \frac{g_2}{\pi^2} - \frac{g_1}{\pi^2} = \frac{1}{\pi^2}(g_2 - g_1)$$

$$= \frac{9.8 - 9.75}{10} = \frac{0.05}{10} = 0.005\ m$$

Ans :	Increase in length of pendulum is 0.005m

Problem for Practice

1. The period of simple pendulum is found to increase by 50% when the length of the pendulum is increased 0.6 m. Calculate the initial length and initial period of oscillation at a place where $g = 9.8\ m/s^2$

Ans: 0.48m, 1.391 sec

2. When the length of a simple pendulum is increased by 44 cm. the period changes by 20%. Find the original length of the pendulum

Ans:1m

3. A clock regulated by second pendulum, keep correct time. During summer. length of pendulum increases 1.005m. How much will the clock gain of loose in one day? ($g = 9.8\ m/s^2$ and $\pi = 3.142$)

Ans: 515.3 sec

4. A simple pendulum of length 1 m and mass 10 g oscillates freely with amplitude 2 cm. Finde its potential energy (P.E) at the extreme position.

Ans:1.96×10^{-5}J

5. The length of the second's pendulum in a clock is increased to 4 times its initial length. Calculate the number of oscillations completed by the new pendulum in one minute.

Ans:15Hz

5.13 Angular S.H.M. and its Differential Equation

Q.21 Define angular S.H.M

Ans : Angular S.H.M. is defined as the oscillatory motion of a body in which the torque for angular acceleration is directly proportional to the angular displacement and its direction is opposite to that of angular displacement.

$\tau \alpha - \theta$

$\therefore$ $\tau = -c\theta$

Where c is constant of proportionality

Q.22 Show the the motion performed by a metallic disc hanging froma rigid support when twisted slightly is an angular S.H.M.

OR

Obtain or differential equation for angular SHM

Ans:

i. The metalic disc when twisted, performs an oscillatory motion for which the restoring torque acting upon it, for angular displacement is θ

$\tau \propto -\theta$

$\therefore$ $\tau = -c\theta$ ...(1)

The constant of proportionality (c) is the restoring torque per unit angular displacement.

ii. If I is the moment of inertia of the disc, the torque acting on the disc is given by,

$\tau = I\alpha$ ...(2)

Where, α is the angular acceleration.

iv. From equation (1) and (2),

$I\alpha = -c\theta$ ... (3)

$\therefore$ $I\frac{d^2\theta}{dt^2} + c\theta = 0$ ...$\left(\because \alpha = \frac{d^2\theta}{dt^2}\right)$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{c\theta}{I} = 0}$$

This is the differential equation for angular S.H.M.

v. From equation (3), the angular acceleration 'α'

is given by,

$$\alpha = -\frac{c\theta}{I}$$

Since c and I are constants, the angular acceleration α is directly proportional to θ and its direction is opposite to that of the angular displacement. Hence, this oscillatory motion is angular S.H.M.

★ Q.23 Prove that under certain conditions a magnet vibrating in uniform magnetic field performs angular SHM. Hence Obtain the expression for the period of a magnet vibrating in a uniform magnetic field and performing S.H.M.

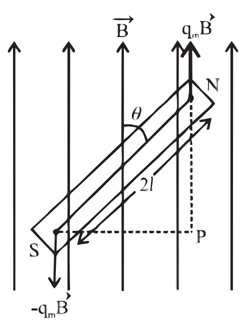
Ans :

i. If a bar magnet a freely suspended in the plane of a uniform magnetic field, it remains in equilibrium with its axis parallel to the direction of the field.

ii. If it is given a small angular displacement θ about an axis passing through its centre, perpendicular to itself and to the field and released, it performs angular oscillations.

iii. Let μ be the magnetic dipole moment and B be the magnetic field.

iv. In the deflected position, a restoring torque acts on the magnet that tends to bring it back to its equilibrium position.



v. The magnitude of this torque is $\tau = \mu B \sin\theta$

If θ is small, $\sin\theta \approx \theta$

$\therefore$ $\tau = \mu B\theta$

vi. For clockwise angular displacement θ, the restoring torque is in the anticlockwise direction.

$\therefore$ $\tau = -\mu B\theta$ ... (1)

$\tau = I\alpha$ ... (2)

vii. Also where, I is the moment of inertia of the bar magnet and α is its angular acceleration.

viii. From (1) and (2) we get,

$I\alpha = -\mu B\theta$

$\therefore$ $$\alpha = -\left(\frac{\mu B}{I}\right)\theta \quad ...(3)$$

Since μ, B and I are constants, equation (3) shows that angular acceleration is directly proportional to the angular displacement and directed opposite to the angular displacement. Hence the magnet performs angular S.H.M.

ix. The period of vibration of the magnet is given by,

$$T = \frac{2\pi}{\sqrt{\text{angular acceleration per unit angular displacement}}}$$

$$= \frac{2\pi}{\sqrt{\alpha / \theta}}$$

Thus, by considering magnitude of angular acceleration (α) from equation (3) We get,

$$T = 2\pi\sqrt{\frac{I}{\mu B}}$$

Type - VII

Numerical based on angular S.H.M.

Formulae Used

1. $T = 2\pi\sqrt{\frac{I}{\mu B}}$

2. $\frac{d^2\theta}{dt^2} + \frac{c}{I}\theta = 0$

3. $\tau = -c\theta$

★ 1) A bar magnet of mass 120 g, in the form of a rectangular parallelepiped, has dimensions *l* = 40 mm, b = 10 mm and h=80mm. With the dimension 'h' vertical, the magnet performs angular oscilations in the plane of a magnetic moment is 3.4 Am², determine the influencing magnetic field.

Data: m = 120 g = 0.12 kg,
$l = 40$ mm $= 40\times10^{-3}$ m,
b = 10 mm $= 10\times10^{-3}$ m,
h = 80 mm $= 8 \times 10^{-2}$ m, T = πs, μ=3.4Am²

To find: Magnetic field (B)

Formulae: i. $I = m\left(\frac{l^2 + b^2}{12}\right)$

ii. $T = 2\pi\sqrt{\frac{I}{\mu B}}$

Solution:

i. $I = m\left(\frac{l^2 + b^2}{12}\right)$

$I = 0.01\left(\frac{1600 + 100}{12}\right) \times 10^{-6}$

$= 1.7 \times 10^{-5}\ Am^2$

ii. $T = 2\pi\sqrt{\frac{I}{\mu B}}$

$\therefore \quad \pi = 2\pi\sqrt{\frac{I}{\mu B}}$

$\therefore \quad 1 = 4 \times \frac{I}{\mu B}$

$\therefore \quad B = \frac{4I}{\mu} = \frac{4 \times 1.7 \times 10^{-5}}{3.4} = 2 \times 10^{-5}\ T$

Ans :	The influencing magnetic field is of strength $2 \times 10^{-5} Wbm^{-2}$ or T

★ **2) Two magnets with the same dimenesions and mass, but of magnetic moments μ_1=100 Am^{-2} and μ_2= 50Am^2 are jointly suspended in the earth's magnetic field so as to perform angular oscillations in a horizontal plane. When their like poles are joined together, the period if their angular S.H.M. is 5 s. Find the period of their angular S.H.M. when their unlike poles are joined together.**

Data: $\mu_1 = 100\ Am^2,\ \mu_2 = 50\ Am^2,\ T_1 = 5s.$

To find: Time period when unlike poles are joined together(T_2)

Formula: $T = 2\pi\sqrt{\frac{1}{\mu B}}$

Solution:

i. With like poles together, the effective magnetic moment is $(\mu_1 + \mu_2)$.

$T = 2\pi\sqrt{\frac{1}{\mu B}}$

$T_1 = 2\pi\sqrt{\frac{1}{(\mu_1 + \mu_2)B}}$ ....(1)

ii. With unlike poles together, the effective magnetic moment is $(\mu_1 - \mu_2)$

$T = 2\pi\sqrt{\frac{1}{\mu B}}$

$T_2 = 2\pi\sqrt{\frac{1}{(\mu_1 - \mu_2)B}}$ ....(2)

iii. Dividing equation (1) by (2),

$\therefore \quad \frac{T_1}{T_2} = \sqrt{\frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)}} = \sqrt{\frac{100 - 50}{100 + 50}}$

$\therefore \quad \frac{5}{T_2} = \sqrt{\frac{50}{150}} = \sqrt{\frac{1}{3}}$

$\therefore \quad \frac{25}{T^2} = \frac{1}{3}$

$\therefore \quad T_2 = \sqrt{25 \times 3} = \sqrt{75} = 5\sqrt{3}$

$= 5 \times 1.732 = 8.66$ s

Ans :	Period of angular S.H.M. when unlike poles are joined together is 8.66 second.

★ **3) A 20 cm wide thin circular disc of mass 200g is suspended to a rigid support form a thin metallic string. By holding the rim of the disc, the string is twisted through 60° and released. It now performs angular oscillations of period 1 second. Calculate the maximum restoring torque generated in the string under undamped conditions. ($\pi^3 \approx 31$)**

Data: $\theta_m = 60° = \frac{\pi}{3}$, T= 1s,

D = 20cm = 0.2 m,

$\therefore$ R = 10cm = 0.1m,
M = 200g = 0.2 kg

To find: (τ_{max})

Formulae: i. $\omega = \frac{2\pi}{T}$ ii. $I = \frac{MR^2}{2}$

ii. $\omega = \sqrt{\frac{c}{I}}$ iv. $\tau_{max} = c\theta_m$

Solution :

i. $\omega = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi \text{ rads}^{-1}$

ii. $I = \frac{MR^2}{2} = \frac{0.2 \times 0.1^2}{2} = 10^{-3} \text{ kg m}^2$

iii. $\omega = \sqrt{\frac{c}{I}}$ $2\pi = \sqrt{\frac{c}{10^{-3}}}$

$\therefore$ $c = 4\pi^2 \times 10^{-3}$ Nm

iv. $\tau_{max} = c\theta_m$

$\tau_{max} = 4\pi^2 \times 10^{-3} \times \frac{\pi}{3} = \frac{4\times\pi^3}{3} \times 10^{-3}$

$= \frac{4\times 31}{3\times 1000} = 0.04133\text{Nm}$

Ans :	The maximum restoring torque is 0.04133 Nm.

4) Find the number of oscillations performed per minute by a magnet vibrating in the plane of a uniform field of 1.6 × 10^{-5} Wb/m^2. The magnet has moment of inertia 3 × 10^{-6} kg/m^2 and magnetic moment 3 Am^2.

Data: $B = 1.6 \times 10^{-5}$ Wb/m^2, $I = 3\times10^{-6}$ kg/m^2
$\mu = 3\text{Am}^2$

To find: n

Formula: $T = 2\pi\sqrt{\frac{I}{\mu B}}$

$\therefore$ $n = \frac{I}{T}$

$\therefore$ $n = \frac{1}{2\pi}\sqrt{\frac{\mu B}{I}}$

Solution: $n = \frac{1}{2\pi}\sqrt{\frac{\mu B}{I}}$

$= \frac{1}{2\times 3.14} = \sqrt{\frac{\cancel{3}\times 1.6\times 10^{-5}}{\cancel{3}\times 10^{-6}}}$

$= \frac{1}{2\times 3.14}\sqrt{16}$

$= \frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{2}}\times 3.14} = \frac{2}{3.14}\text{Hz}$

$= \frac{2}{3.14}\times 60 \text{ osc / min}$

$= \frac{120}{3.14} \text{ osc / min} = 38.19 \text{ osc/min}$

Ans :	The number of oscillations performed per minute by the magnet is 38.19.

Problem for Practice

1. A bar magnet is oscillatins in the earth's magnetic field with a time period T. What will be the new time period if mass of bag magnet is increased four times

Ans : 2T

2. A magnet has a time period of oscillation T in earth's magnetic field at a place. If it is replaced by another magnet of magnetic moment six times the magnetic moment of the original one. What will be the time period at same place

Ans: $\frac{T}{\sqrt{16}}$

3. Two different magnets are tied together and allowed to vibrate in horizontal plane with their like poles joined together the time period is 5 sec and with unlike poles joined together time period is 15 sec. Find the ratio of their magnetic mements.

Ans: $\frac{5}{4}$

5.14 Damped Oscillations

Q.24 Explain damped S.H.M.

Ans:

i. In the motion of a simple pendulum air drag and the friction at the support oppose the motion of the pendulum. The energy of the system is dissipated continuously. (but, for small damping, the oscillation remain approximately periodic.)

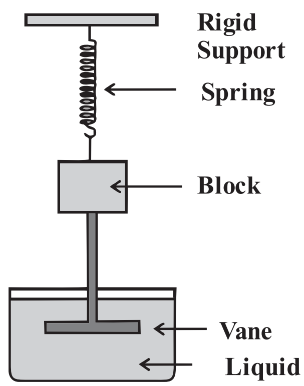
ii. Let's consider an example of spring pendulum oscillates vertically of which angular frequency is given by,

$$\omega = \sqrt{\frac{k}{m}}$$

where, m - mass of block connected to an elastic spring

k- spring constant

iii. In practice the surrounding medium (water) will exert a damping force on the motion of the block so its mechanical energy will decrease as shown in fig.



iv. The damping force depends on the nature of the surrounding medium. The damping force is generally proportional to velocity as per the Stokes' law, and acts opposite to the direction of velocity.

∴ Damping force , $F_d = -bv$

where b depends on viscosity of the medium and the size and shape of the block, etc.

v. When the mass m is attached to the spring and released, the spring will elongate a little and the mass will settle at some height.

∴ Restoring force , $F_s = -kx$,

where x is the displacement of the mass from its equilibrium position.

vi. The total force acting on the mass,

$F = -kx - bv$

∴ $F + kx + bv = 0$

Substituting, $F = m\frac{d^2x}{dt^2}$ and $v = \frac{dx}{dt}$

∴ above eq^n become,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k\ x = 0 \qquad ...(1)$$

This is differential equation for damped oscillation.

vii. **Displacement :** Solving equation (i) under the influence of a damping force we get,

$x = A\, e^{-bt/2m} \cos(\omega' t + \phi) \qquad ...(2)$

where, A is the amplitude

and ω' is the angular frequency of the damped oscillator given by.

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

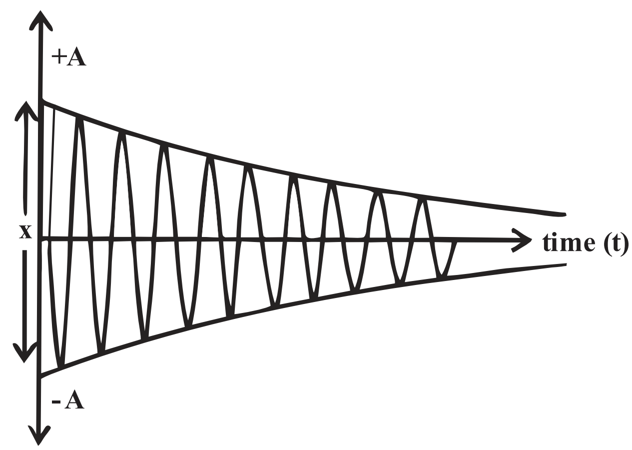
where, period

∴ $T = 2\pi/\omega'$

$$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}$$

but the function $x(t)$ is not strictly periodic because of the factor $e^{-bt/2m}$ which decreases continuously with time.

viii. Equation (2) can be graphically represented as shown in fig. whose amplitude which is $Ae^{-bt/2m}$ gradually decreases with



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