01 Rotational Dynamics

Syllabus

1.1 Introduction

Q.1 What is circular motion ?

Ans. Circular Motion : The motion of particle along the circumference of circle is called circular motion.

Characteristics of Circular Motion

1) It is an accelerated motion : As the direction of velocity changes at every instant, it is an accelerated motion.

2) It is a periodic motion : During the motion, the particle repeats its path along the same trajectory. Thus the motion is periodic.

1.2.1 Kinematics of circular motion

Recap of 11the std.

(a) Circular motion is of two types

i. Uniform Circular Motion (UCM)

In uniform circular motion body (Particle) moves along a circular path with constant speed.

Example: Motion of tip of hands of clock, motion of planets around sun.

ii. Non-Uniform Circular Motion (Non-UCM)

In non uniform circular motion. Particle moves along a circular path with variable speed.

Example: Vertical circular motion.

Terms involved in circular motion.

(i) Radius Vector ($\vec{r}$) :

a. *Vector drawn from centre of circular path to any instantaneous position of the particle performing circular motion is called radius vector.*

b. *S.I. unit is metre*

c. *Dimensions – $[M^0 L^1 T^0]$*

d. *Direction is away from centre*

e. *It is also called Position Vector*

f. *Magnitude of radius vector is equal to radius of circle*

(ii) Angular displacement (θ) :

Angle described by the radius vector in a given time at the centre of a circle.

a. *Formula :* $\theta = \dfrac{\text{Arc length}}{\text{radius}}$; $\theta = \dfrac{s}{r}$

b. *Infinitesimal angular displacement $\vec{d\theta}$ is a vector quantity. Finite angular displacement (θ) is a scalar quantity.*

c. *Direction is given by right hand rule or right handed screw rule.*

Dimensions : $[M^0L^0T^0]$

d. *S. I. unit is radian.*

(iii) Angular Velocity (ω) :

The rate of change of angular displacement with time is called angular velocity.

a. *Instantaneous angular velocity is*

$$\vec{\omega} = \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\vec{\theta}}{dt}$$

b. *Average angular velocity is* $\omega = \frac{\theta}{t}$

c. *S.I unit is radians/sec*

d. *Dimensions* $[M^0 L^0 T^{-1}]$

e. *It is vector given by right hand screw rule.*

(iv) *Angular acceleration (α) :*

The rate of change of angular velocity with time is called angular acceleration.

a. *Instantaneous angular acceleration is*

$$\vec{\alpha} = \lim_{\delta t \to 0} \frac{\delta\omega}{\delta t} = \frac{d\vec{\omega}}{dt}$$

b. *If the angular velocity ω_1 changes to ω_2 in time t, then average angular acceleration is*

$$\alpha = \frac{\omega_2 - \omega_1}{t}$$

c. *S. I. unit is radians/sec²*

d. *Dimensions* $[M^0 L^0 T^{-2}]$

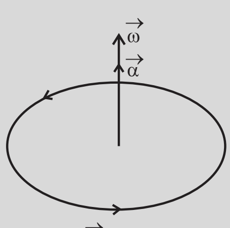
e. *It is vector given by right hand screw rule.*

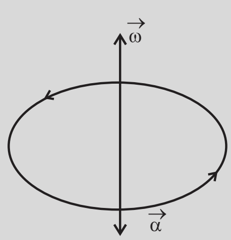
Right Hand Rule

Imagine the axis of rotation to be held in right hand with the fingers curled around it and thumb outstretched. If the curled fingers give the direction of motion of a particle performing circular motion, then the direction of outstretched thumb gives the direction of angular displacement vector.

Note:

Sign conventions for angular displacement and angular velocity:





a. *For anticlockwise circular motion infinitesimal angular displacement and angular velocity are perpendicular to the plane of the circle and directed vertically upward and considered as **positive**.*

b. *For clockwise circular motion infinitesimal angular displacement and angular velocity are perpendicular to the plane of the circle and directed vertically downward and considered as **negative**.*

i. *Positive angular acceleration:*

When angular velocity increases, angular acceleration will have the same direction as angular velocity. This is an example of positive angular acceleration,

ii. *Negative angular acceleration:*

When angular velocity decreases, angular acceleration will have a direction opposite to that of angular velocity. This is an example of negative angular acceleration

(v) *Period (T) :*

a. *Time taken by the particle performing uniform circular motion to complete one revolution.*

b. *During periodic time the particle covers a distance equal to the circumference of the circle*

$$Period = \frac{Circumference}{Linear\ Velocity}$$

$$T = \frac{2\pi r}{v}$$

c. *During periodic time, the particle has an angular displacement 2π radians.*

$$Period = \frac{Angular\ displacement}{Angular\ Velocity}$$

$$T = \frac{2\pi}{\omega}$$

d. *S.I. unit is seconds.*

(vi) Frequency (n) :

a. *It is number of revolution completed by a particle performing uniform circular motion in unit time.*

b. $n = \frac{1}{T}$

c. $n = \frac{\omega}{2\pi} = \frac{v}{2\pi r}$

$\therefore \omega = 2\pi n$

d. *S. I unit is hertz.*

Note:

Relation between Linear and Angular velocity

$v = r\omega$

In Vector Form

$\vec{v} = \vec{\omega} \times \vec{r}$

(d) Acceleration in circular motion

1) *The acceleration responsible for circular motion is centripetal or radial acceleration*

$\vec{a} = -\omega^2 \vec{r}$

negative sign indicates $\vec{a}$ and $\vec{r}$ are opposite in direction. As $\vec{r}$ is directed away from centre.

Therefore $\vec{a}$ is directed towards centre. Hence such type of acceleration is called radial or centripetal acceleration.

2. *For particle performing circular motion, the linear acceleration is tangential to the path, as linear velocity of path changes tangentially such acceleration is called tangential acceleration.*

$a_T = r\alpha$

In UCM, ω = constant

$\therefore \alpha = 0$

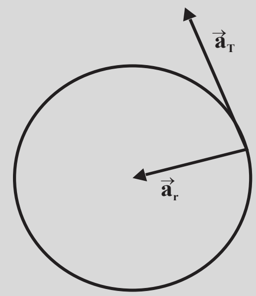
$\therefore a_T = 0$

$\therefore$ *acceleration in UCM is Centripetal acceleration*

$a_c = \omega^2 r = \frac{V^2}{r}$

In Non UCM,

$a = \sqrt{a_T^2 + a_c^2}$



MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 1)

Basic of Circular Motion

1. A particle moves in a circle of radius 5cm with constant speed and time period 0.2π second. The acceleration of particle is

a. 15 m/s^2 b. 2.5 m/s^2
c. 36 m/s^2 d. 5 m/s^2

2. A body executing uniform circular motion has at any instant its velocity vector and acceleration vector

a. along the same direction
b. in opposite direction
c. normal to each other
d. not related to each other

3. A stone tied to the end of a string 1m long is whirled in a horizontal circle with a constant speed if the stone makes 22 revolutions in 44 second, what is the magnitude and direction of acceleration of the stone ?

a. $\pi^2/4$ m/s^2 and direction along the radius towards the centre
b. π^2 m/s^2 and direction along the radius and away from centre
c. π^2 m/s^2 and direction along the radius towards the centre
d. π^2 m/s^2 and direction along the tangent to the circle.

4. A car runs at a constant speed on a circular track of radius 100m, taking 62.8 second in every circular loop.
The average velocity and average speed for each circular loop respectively is

a. 0, 10m/s b. 10m/s, 10m/s

c. 10m/s, 0 d. 0, 0

5. If the equation for the displacement of a particle moving on a circular path is given by $\theta = 2t^3 + 0.5$ where θ is in radian and t in second, then the angular velocity of the particle of t = 2 sec is

a. 8 rad/s b. 12 rad/s
c. 24 rad/s d. 36 rad/s

6. Two racing cars of masses m_1 and m_2 are moving in circles of radius r_1 and r_2 respectively. Their speeds are such that each makes complete circle in same time 't'. The ratio of angular speeds of the first to the second car is

a. $r_1 : r_2$ b. $m_1 : m_2$
c. 1 : 1 d. $m_1m_2 : r_1r_2$

7. What is the value of linear velocity if $\vec{\omega} = 3\hat{i} - 4\hat{j} + \hat{k}$ and $\vec{\omega} = 5\hat{i} - 6\hat{j} + 6\hat{k}$

a. $4\hat{i} - 13\hat{j} + 6\hat{k}$ b. $18\hat{i} + 13\hat{j} - 2\hat{k}$
c. $6\hat{i} + 2\hat{j} - 3\hat{k}$ d. $6\hat{i} - 2\hat{j} + 8\hat{k}$

8. The magnitude of displacement of a particle moving in a circle of radius a with constant angular speed ω varies with time t is

a. $2a \sin \omega t$ b. $2a \sin \omega t/2$
c. $2a \cos\omega t$ d. $2a \cos\omega t/2$

9. For a particle moving in a circle of radius 0.4 m at the instant its angular velocity is 2 rad/s^2 and its angular acceleration is 5 rad/s^2. Then its linear acceleration is

a. 2 m/s^2 b. 2.6 m/s^2
c. 3.8 m/s^2 d. 5 m/s^2

10. A car is circulating on the path of radius r and at any time its velocity is V and rate of increase of velocity is a. The resultant acceleration of car will be

a. $\sqrt{\frac{V^2}{a^2} + r}$ b. $\sqrt{\frac{V^2}{r} + a}$

c. $\sqrt{\frac{V^4}{r^2} + a^2}$ d. $\left(\frac{V^2}{r} + a\right)$

Try Yourself

11. The ratio of angular speeds of minute hand and hour hand of a watch is

a. 1 : 12 b. 6 : 1
c. 12 : 1 d. 1 : 6

12. A particle *p* is moving in a circle of radius 'a' with a uniform speed *v*. *C* is the centre of the circle and AB is a diameter. When passing through *B* the angular velocity of about and are in the ratio

a. 1 : 1 b. 1 : 2
c. 2 : 1 d. 4 : 1

13. The average acceleration vector for a particle having a uniform circular motion is

a. A constant vector of magnitude $\frac{v^2}{r}$

b. A vector of magnitude $\frac{v^2}{r}$ directed normal to the plane of the given uniform circular motion

c. Equal to the instantaneous acceleration vector at the start of the motion

d. A null vector

14. A particle comes round a circle of radius 1 *m* once. The time taken by it is 10 *sec*. The average velocity of motion is

a. 0.2 b. $2\pi m/s$
c. 2 d. Zero

15. If a particle covers half the circle of radius R with constant speed then

a. Momentum change is mvr
b. Change in K.E. is 1/2 mv^2
c. Change in K.E. is mv^2
d. Change in K.E. is zero

16. The angular velocity of a wheel is 70 *rad/sec*. If the radius of the wheel is 0.5 *m*, then linear velocity of the wheel is

a. 70 *m/s* b. 35 *m/s*
c. 30 *m/s* d. 20 *m/s*

17. A scooter is going round a circular road of radius 100 *m* at a speed of 10 *m/s*. The angular speed of the scooter will be

a. 0.01 *rad/s* b. 0.1 *rad/s*
c. 1 *rad/s* d. 10 *rad/s*

18. A car of mass 800 kg moves on a circular track of radius 40 m. If the coefficient of friction is 0.5, then maximum velocity with which the car can move is

a. 7 *m/s* b. 14 *m/s*
c. 8 *m/s* d. 12 *m/s*

Type - I

Problems based on Kinematic equation.

Formula used

1. Relation between linear and angular term

	Linear	Angular
Displacement	s	θ
initial velocity	u	ω_0
final velocity	v	ω
acceleration	a	α
1st kinematic equation	$v = u + at$	$\omega = \omega_0 + \alpha t$
2nd kinematic equation	$s = ut + \frac{1}{2}at^2$	$\theta = \omega_t + \frac{1}{2}\alpha t^2$
3rd kinematic equation	$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

2. Angular acceleration is rate of change of angular velocity w.r.t. time

$$\alpha = \frac{\omega_2 - \omega_1}{t}$$

$$\alpha = \frac{2\pi n_2 - 2\pi n_1}{t}$$

$$\boxed{\alpha = 2\pi\left(\frac{n_2 - n_1}{t}\right)}$$

★1) A fan is rotating at 90 rpm . It is then switched off. It stops after 21 revolutions. Calculate the time taken by it to stop. assuming that the frictional torque is constant.

Data: $n_0 = 90 \text{ rpm} = \frac{90}{60} \text{ rps} = \frac{3}{2} \text{ Hz} = 1.5 \text{ Hz}$

$N = 21, \omega = 0$

To find: time taken by fan to stop (t)

Formula: i. $\theta = 2\pi N$

ii. $\omega = 2\pi n$

iii. $\omega = \omega_0 + \alpha t$

iv. $\omega^2 = \omega_0^2 + 2\alpha\theta$

Solution:

i. $\theta = 2\pi N = 2\pi \times 21 = 42\pi^c$

ii. $\omega_0 = 2\pi \times n_0 = 2\pi \times \frac{3}{2} = 3\pi \text{ rad/s}$

iii. From 1st Kinematic equation

$\omega = \omega_0 + \alpha t$

$\therefore \quad \alpha = \frac{\omega - \omega_0}{t}$ ...(1)

iv. From 3rd kinematic equation

$\omega^2 - \omega_0^2 + 2\alpha\theta$

$\therefore \quad \alpha = \frac{\omega^2 - \omega_0^2}{2\theta}$ ...(2)

v. As frictional torque is constant, retardation is constant, α = constant

$\therefore$ Equating eq. (1) and (2), we get

$$\frac{\omega^2 - \omega_0^2}{2\theta} = \frac{\omega - \omega_0}{t}$$

$$t = \frac{(\omega - \omega_0) \times 2\theta}{\omega^2 - \omega_0^2} = \frac{(\omega - \omega_0) \times 2\theta}{(\omega + \omega_0)(\omega - \omega_0)}$$

$$\therefore \quad t = \frac{2\theta}{\omega + \omega_0} = \frac{2 \times 42\pi}{0 + 3\pi} = \frac{84\pi}{3\pi} = \frac{28}{1}$$

$\therefore$ t = 28 sec.

Ans : Time taken by fan to stop is 28 second.

2) A flywheel is rotating at the rate of 100 rpm and slows down at a constant rate of 1 rad/s^2 calculate the time required to stop the flywheel and the number of rotation made by the flywheel before coming to rest ?

Data: $n_0 = 100 \text{ rpm} = \frac{100}{60} = \frac{5}{3} \text{Hz}$

$\alpha = -1\ rad/s^2$, $\omega = 0\ rad/s$

To find: i. t ii. N

Formula:
i. $\omega = 2\pi n$
ii. $\omega = \omega_0 + \alpha t$
iii. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
iv. $\theta = 2\pi N$

Solution:

i. $\omega_0 = 2\pi n_0 = 2\pi \times \frac{5}{3} = \frac{10\pi}{3}\ rad/s$

ii. $\omega = \omega_0 + \alpha t$

$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 10\pi/3}{-1}$

$= \frac{10\pi}{3} = \frac{10 \times 3.14}{3} = 10.47\ s.$

iii. $\theta = \omega_o t + \frac{1}{2}\alpha t^2$

$= \frac{10\pi}{3} \times \frac{10\pi}{3} + \frac{1}{2}(-1) \times \left(\frac{10\pi}{3}\right)^2$

$= \left(\frac{10\pi}{3}\right)^2 \left[1 - \frac{1}{2}\right] = \frac{100 \times \pi^2}{9} \times \frac{1}{2}$

$\theta = \frac{100\pi^2}{18}$

iv. $\theta = 2\pi N$

$N = \frac{\theta}{2\pi} = \frac{100\pi^2}{18 \times 2\pi} = \frac{100\pi}{18 \times 2} = \frac{100 \times 3.14}{36}$

= Anti log

log N		log D	
log100	2.0000	log36	1.5563
log3.14	+ 0.4969		
	2.4969		

$= \text{Anti log}\begin{bmatrix} 2.4969 \\ -1.5563 \\ \hline 0.9406 \end{bmatrix}$

N = 8.722

Ans :	10.47 s is required to stop the fly wheel and before stopping flywheel makes 8.722 rotation.

★ **3) Some how, an ant is stuck to the rim of a bicycle wheel of diameter 1 m. While the bicycle is on a central stand, the wheel is set into rotation and it attains the frequency of 2 rev/s in 10 seconds, with uniform angular acceleration. Calculate**
i. Number of revolutions completed by the ant in these 10 seconds
ii. Time taken by it for first complete revolution and the last complete revolution.

Data: D = 1m, r = 0.5 m, $n_0 = 0$ Hz,
n = 2 Hz, t = 10 s

To find:
i. Number of revolution (N) in 10 seconds
ii. time taken for first revolution (t_1)
iii. time taken for last revolution (t_{10})

Formula:
i. $\omega = 2\pi n$
ii. $\omega = \omega_0 + \alpha t$
iii. $\theta = w_0 t + \frac{1}{2}\alpha t^2$
iv. $N = \frac{\theta}{2\pi}$

Solution:

i. $\omega = 2\pi n = 2\pi \times 2 = 4\pi\ rad/s$

$\omega_0 = 2\pi n_0 = 0\ rad/s$

ii. $\omega = \omega_0 + \alpha t$

$\therefore$ $\alpha = \frac{\omega - \omega_0}{t} = \frac{4\pi - 0}{10} = \frac{2\pi}{5}\ rad/s^2$

iii. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\theta = 0 + \frac{1}{2} \times \frac{2\pi}{5} \times (10)^2$

$= \frac{2\pi \times 100}{10} = 20\pi\ rad$

iv. $N = \frac{\theta}{2\pi} = \frac{20\pi}{2\pi} = 10\ rev.$

v. To calculate time (t_1) taken for first revolution

$\theta = 2\pi$, $\omega_0 = 0\ rad/s$

$\therefore$ $\theta = \omega_0 t_1 + \frac{1}{2}\alpha\, t_1^2$

$$2\pi = 0 + \frac{1}{2} \times \frac{2\pi}{5} \times t_1^2$$

$$t_1^2 = \frac{10\pi}{\pi}$$

$\therefore$ $t_1 = \sqrt{10}$ sec

vi. To calculate time (t_{10}) taken in last revolution

Let us First calculate, time taken for 9 revolution.

$\theta = 18\pi$, $\omega_0 = 0$ rad / s

$\therefore$ $\theta = \omega_0 t_9 + \frac{1}{2} \alpha t_9^2$

$$18\pi = 0 + \frac{1}{2} \times \frac{2\pi}{5} \times t_9^2$$

$$t_9^2 = \frac{18 \times 5\pi}{\pi}$$

$$t_9^2 = 90$$

$$t_9 = \sqrt{90} = 9.4868 \text{ sec.}$$

As we know wheel rotates for 10 sec.

$t_{10} = 10 - 9.4868$

$= 0.5132$ sec.

Ans : i. Number of revolution completed by ant is 10
ii. time taken for 1st revolution is $\sqrt{10}$ sec.
iii. time taken for last revolution is 0.5132 sec.

Problem for Practice

1. The number of revolution made by a flywheel change from 300 rpm to 1500 rpm in 10 second. Calculate angular acceleration assuming it to be uniform. Also calculate the number of revolutions made during the time.

Ans. 4π rad/s^2, 150 revolutions

2. When a constant torque is applied, a wheel is turned from rest through 200 radian in 10 s. What is angular acceleration. If the same torque continued to act what is the angular velocity of the wheel after 15 second from the start.

Ans. 60 rad/s

3. The frequency of revolution of a particle performing circular motion changes from 60 r.p.m to 180 r.p.m in 20 seconds. Calculate the angular acceleration of the particle.

Ans : 0.6284 rad/s^2

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 2)

Angular Kinematic equation

1. A wheel has angular acceleration of 3.0 rad/sec^2 and an initial angular speed of 2.00 rad/sec. In a time of 2 sec it has rotated through an angle (in radian) of

a. 10 b. 12
c. 4 d. 6

2. A wheel is at rest. Its angular velocity increases uniformly and becomes 60 rad/sec after 5 sec. The total angular displacement is

a. 600 rad b. 75 rad
c. 300 rad d. 150 rad

3. A wheel initially at rest, is rotated with a uniform angular acceleration. The wheel rotates through an angle θ_1 in first one second and through an additional angle θ_2 in the next one second. The ratio $\frac{\theta_2}{\theta_1}$ is

a. 4 b. 2
b. 3 d. 1

Try Yourself

4. A car is moving at a speed of 72 km/hr. the diameter of its wheels is 0.5 m. If the wheels are stopped in 20 rotations by applying brakes, then angular retardation produced by the brakes is

a. – 25.5 rad/s^2 b. – 29.5 rad/s^2
c. – 33.5 rad/s^2 d. – 45.5 rad/s^2

5. A wheel is rotating at 900 r.p.m. about its axis. When the power is cut-off, it comes to rest in 1 minute. The angular retardation in radian/s^2 is

a. $\pi/2$ b. $\pi/4$ c. $\pi/6$ d. $\pi/8$

6. A strap is passing over a wheel of radius 30 cm. During the time the wheel moving with

initial constant velocity of 2 rev/sec. comes to rest the strap covers a distance of 25 m. The deceleration of the wheel in rad/s^2 is

a. 0.94 b. 1.2 c. 2.0 d. 2.5

7. When a ceiling fan is switched off, its angular velocity falls to half while it makes 36 rotations. How many more rotations will it make before coming to rest (Assume uniform angular retardation)

a. 36 b. 24 c.18 d. 12

8. Let $\vec{A}$ be a unit vector along the axis of rotation of a purely rotating body and $\vec{B}$ be a unit vector along the velocity of a particle P of the body away from the axis. The value of $\vec{A}.\vec{B}$ is

a. 1 b. – 1
c. 0 d. None of these

Q.2 Explain the concept of centripetal force and centrifugal force

Ans: Centripetal force :

In uniform circular motion the constant force acting on the particle which is along the radius and is directed towards the centre. This force is called centripetal force.

Explanation :

i. Centripetal force is necessary for maintaining circular motion.

ii. According to Newtons 2nd law of motion.

$$\begin{pmatrix}\text{Centripetal}\\\text{force}\end{pmatrix} = (\text{mass}) \times \begin{pmatrix}\text{centripetal}\\\text{acceleration}\end{pmatrix}$$

$$= \frac{mv^2}{r} = mr\omega^2$$

iii. Centripetal force is due to known interaction. Therefore, it is a real force.

iv. It is non-conservative force because in circular motion, at every point linear displacement is perpendicular to linear velocity at that point.

Examples :

a. If an object is tied to a string is revolved uniformly in a horizontal circle, the centripetal force is due to the tension imparted to the string by the hand.

b. When the satellite is revolving in circular orbit round the earth, the centripetal force is due to the gravitational force of attraction between the satellite and the earth.

c. In an atom, an electron revolves in a circular orbit round the nucleus. The centripetal force is due to the electrostatic force of attraction between the positively charged nucleus and negatively charged electron.

Centrifugal force:

When the frame of reference of the particle is accelerated, the pseudo-force experienced by the particle performing uniform circular motion is along the radius but away from the centre. It is called centrifugal force.

Explanation :

i. Centrifugal force is a pseudo- force as it does not arise due to any real interaction but is experienced only due to accelerated frame of reference for the particle.

ii. It acts along the radius and away from the centre.

iii. The centrifugal force has the same magnitude as that of the centripetal force. But, its direction is opposite to that of the centripetal force.

iv. Magnitude of the centrifugal force is $\frac{mv^2}{r}$ or $mr\omega^2$

Examples :

a. When a car moving along a horizontal road takes a turn, a person in the car experiences a pull in the outward direction.

b. The coin placed slightly away from the centre of a rotating gramophone disc slips towards the edge of the disc.

c. A cyclist moving fast along a curved road has to lean inwards to keep his balance as the centrifugal force pushes him outwards.

Q.3 Distinguish between Centripetal Force and Centrifugal Force

Ans :

	Centripetal Force	Centrifugal Force
i	It is force action along radius towards centre	It is force acting along radius and away from centre.
ii.	It is real force	It is pseudo force
iii.	It exist in both inertial frame of reference and non-inertial frame of reference	It exist in non-inertial frame of reference
iv.	It is required for circular motion	It is produced due to circular motion

Type - II
Numerical on Centripetal force and tension in string

1) A body of mass 2 kg is tied to the end of a string of length 1.5 m and revolved uniformly about the other end in a horizontal circle. If it makes 300 rev/min. Find centripetal force acting upon the body. (Take $\pi^2 = 10$)

Data: $m = 2$ kg, $r = 1.5$ m,

$$n = 300 \text{ rpm} = \frac{300}{60} = 5 \text{ Hz}$$

To find: Centripetal Force (F)

Formula: i. $\omega = 2\pi n$
ii. $F = mr\omega^2$

Solution:

i. $\omega = 2\pi n = 2\pi \times 5 = (10\pi)^2$ rad / s

ii. $F = mr\omega^2$

$= 2 \times 1.5 \times (10\pi)^2$

$= 3 \times 100 \times \pi^2 = 300 \times 9.85$

$F = 2957.9$ N

Ans : Centripetal force is 2957.9 N

2) A certain string breaks under a tension of 45 kg-wt. A mass of 100gm is attached to one end of a piece of this string 5 m long and rotated in horizontal circle. Find the greatest number of revolution which the string can make without breaking.

Data: $T_B = 45 \text{ kg} - \text{wt} = 45 \times 9.8$ N

$r = 5$ m, $m = 100\text{g} = 10^{-1}$ kg.

To find: n_{max} (maximum revolution)

Formula: Breaking Tension = $mr\, w^2_{max}$

$$T_B = mr(2\pi n_{max})^2$$

$$T_B = mr \times 4\pi^2 n^2_{max}$$

$$n^2_{max} = \frac{T_B}{mr \times 4\pi^2}$$

Solution:

$$n^2_{max} = \frac{T_B}{mr \times 4\pi^2}$$

$$n^2_{max} = \frac{45 \times 9.8}{10^{-1} \times 5 \times 4 \times (3.14)^2}$$

$$n^2_{max} = \frac{45 \times 98}{20 \times (3.14)^2}$$

= Anti log

log N		log D	
log 45	1.6532	log 20	1.3010
log 98	+ 1.9912	log 3.14	0.4969
	3.6444	log 3.14	0.4969
			2.2948

$$= \text{Antilog}\begin{bmatrix} 3.6444 \\ -2.2948 \\ 1.3496 \end{bmatrix}$$

$$n^2_{max} = 22.37$$

$$n_{max} = \sqrt{22.37}$$

$$n_{max} = 4.729 \text{ Hz}$$

Ans : A string can make maximum 4.729 revolutions per sec.

Problem for Practice

1. A body of mass 0.25kg is tied to one end of a string and rotated uniformly about the other end in a horizontal circle of one metre radius at 80 r.p.m. Calculate its (1) angular velocity (2) linear velocity and (3) period. Also find the force acting upon the body and the tension in the string.

Ans: $\frac{8\pi}{3}$ **rad / s,** $\frac{8\pi}{3}$ **m /s,**
0.75 s., 17.5 N, 17.5 N

2. A one kg mass is whirled in a horizontal circle at the end of a string 0.5m long, the other end of the string being fixed. The breaking tension in the string is 50 N. Find the greatest speed that can be given to the mass.

Ans: 5 m /s

3. A 0.5 kg mass is tied to one end of a string and rotated in a horizontal circle of 1.25 m radius about the other end. What is the tension in the string if the period of revolution is 5 s? What is the maximum speed of rotation and the corresponding period if the string can withstand a maximum tension of 150 N?

And : 59N;19.36 m/s;0.16s

Type - III
Numerical based on Centripetal Force
[coins wale sums]

Formula:

1. Centripetal Force

$$F = \frac{mv^2}{r} = mr\omega^2$$

2. When coin is placed on rotating disc, the necessary centripetal force is provided by frictional force between coin and surface of disc

Centripetal force = Frictional force

3. To calculate maximum speed of coin

$$\frac{mv^2}{r} = \mu mg$$

$\therefore$ $v^2 = \mu r g$

$\therefore$ $\boxed{v = \sqrt{\mu r g}}$

4. To calculate maximum angular speed of coin

$$mr\omega^2 = \mu_s m g$$

$$\omega^2 = \frac{\mu g}{r}$$

$$\boxed{\omega = \sqrt{\frac{\mu g}{r}}}$$

★1) Coefficient of static friction between a coin and a gramophone disc is 0.5. Radius of the disc is 8 cm. Initially the centre of the coin is π cm away from the centre of the disc. At what minimum frequency will it start slipping from there? By what factor will the answer change if the coin is almost at the rim? (use g = π^2 m/s^2)

[Ans: 2 rev/s, $n_2 = 0.63\ n_1$]

Data: $\mu = 0.5$

$R = 8\text{ cm} = 8 \times 10^{-2}\text{ m}$

$r = \pi\text{ cm} = \pi \times 10^{-2}\text{ m}$

To find: i. Minimum speed at which coin start slipping (n)

ii. Factor by which frequency of coin changes if placed at rim $\left(\frac{n}{n^1}\right)$

Formula:

Centripetal Force = Frictional Force

$$mr\omega^2 = \mu mg$$

$$\omega^2 = \frac{\mu g}{r}$$

$$(2\pi n)^2 = \frac{\mu g}{r}$$

$$4\pi^2 n^2 = \frac{\mu g}{r}$$

$$n^2 = \frac{\mu g}{4\pi^2 r}$$

$$n = \sqrt{\frac{\mu g}{4\pi^2 r}}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{\mu g}{r}}$$

Solution:

i. Minimum frequency at which coint start slipping

$$n = \frac{1}{2\pi}\sqrt{\frac{\mu g}{r}} \quad ...(1)$$

$$= \frac{1}{2\times\pi}\sqrt{\frac{0.5\times\pi^2}{\pi\times10^{-2}}} = \frac{\pi}{2\pi\times10^{-1}}\sqrt{\frac{1}{2\pi}}$$

$$= \frac{10}{2}\sqrt{\frac{1}{6.28}} = 5\times\sqrt{0.1542}$$

$$= 5\times\sqrt{15.92\times10^{-2}} = 5\times3.99\times10^{-1}$$

$= 19.95 \times 10^{-1} = 1.995$ Hz

ii. When coin is at rim

$r' = 8 \times 10^{-2}$ m

$$n' = \frac{1}{2\pi}\sqrt{\frac{\mu g}{r^1}} \quad ...(2)$$

Now (2) ÷ (1)

$$\frac{n'}{n} = \sqrt{\frac{r}{r'}}$$

$$\frac{n'}{n} = \sqrt{\frac{\pi\times10^{-2}}{8\times10^{-2}}} = \sqrt{\frac{3.14}{8}}$$

$$\frac{n'}{n} = \sqrt{0.3925} = 0.6265$$

$n^1 = 0.6265$ n

Ans: i. Minimum frequency at which coin starts slipping is 1.995 Hz.
ii. Factors by which frequency changes if coin is placed at rim is 0.6265.

2) A coin just remains on the disc rotating at steady rate of 180 rpm if kept at a distance of 2 cm from the axis of rotation. Find coefficient of friction between coin and the disc. (Take g = 10 m/s²)

Data: $n = 180 \text{ rpm} = \frac{180}{60}\text{Hz} = 3$ Hz

$r = 2 \text{ cm} = 2 \times 10^{-2}$ m

To find: μ

Formula: Centripetal Force = Frictional Force

$$\cancel{m}r\omega^2 = \mu\cancel{m}g$$

$$r\times(2\pi n)^2 = \mu g$$

$$\mu = \frac{4\pi^2 n^2 r}{g}$$

Solution: $$\mu = \frac{4\times(3.14)^2\times(3)^2\times2\times10^{-2}}{10}$$

$$= 4\times(3.14)^2\times9\times2\times10^{-3}$$

$$= 709.8\times10^{-3} = 0.7098$$

Ans: The coefficient of friction between coin and disc is 0.7098.

Problem for Practice

1. A small coin is placed on a turntable at a distance 7 cm from its axis of rotation. The coin begins to slide just as the turntable reaches a speed of 60 rpm. Calculate the rate of rotation for which sliding would commence if (a) the coin were placed 12 cm from the axis. (b) the coin is placed in the original position with another similar coin stuck on top of it.

Ans : 45.82 rpm; 60 rpm

2. A coin kept on a horizontal rotating disc has its centre at a distance of 0.1 m from the axis of the rotating disc. If coefficient of friction between coin and disc is 0.25. Find the angular speed of disc at which the coin would be about to slip off.

Ans : 5rad/sec

3. A coin kept at a distance of 5 cm from the centre of a turtable of radius 1.5 m just begins to slip when the turtable rotates at a speed of 90 rpm. Calculate coefficient of static friction between coin and the tum table.

Ans : $\mu_s = 0.457$

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 3)

Centripetal Force.

1. A stone of mass m is tied to a string of length *l* and rotated in a circle with a constant speed v. If the string is released, the stone flies
a. Radially outward
b. Radially inward
c. Tangentially outward
d. With an acceleration $\frac{mv^2}{l}$

2. Two masses M and m are attached to a vertical axis by weightless threads of combined length l. They are set in rotational motion in a horizontal plane about this axis with constant angular velocity ω. If the tensions in the threads are the same during motion, the distance of from the axis is

a. $\frac{Ml}{M+m}$ b. $\frac{ml}{M+m}$

c. $\frac{M+m}{M}l$ d. $\frac{M+m}{m}l$

3. A string breaks if its tension exceeds 10 *newtons*. A stone of mass 250 *gm* tied to this string of length 10 *cm* is rotated in a horizontal circle. The maximum angular velocity of rotation can be

a. 20 rad/s b. 40 rad/s

c. 100 rad/s d. 200 rad/s

4. A 500 kg car takes a round turn of radius 50 m with a velocity of 36 km/hr. The centripetal force is

a. 250 N b. 750 N

c. 1000 N d. 1200 N

5. A ball of mass 0.25 kg attached to the end of a string of length 1.96 m is moving in a horizontal circle. The string will break if the tension is more than 25 N. What is the maximum speed with which the ball can be moved

a. 14 m/s b. 3 m/s

c. 3.92 m/s d. 5 m/s

6. A body of mass 5 kg is moving in a circle of radius 1 m with an angular velocity of 2 radian/sec. The centripetal force is

a. 10 N b. 20 N

c. 30 N d. 40 N

7. If a particle of mass m is moving in a horizontal circle of radius r with a centripetal force $(-k/r^2)$ the total energy is

a. $-\frac{k}{2r}$ b. $-\frac{k}{r}$

c. $-\frac{2k}{r}$ d. $-\frac{4k}{r}$

Try Yourself

8. A mass 2 kg describes a circle of radius 1 m on a smooth horizontal table at a uniform speed. It it is joined to the centre of the circle by a string, which can just withstand 32 N, then the greatest number of revolutions per minute, performed by the mass would be

a. 38 b. 4

c. 76 d. 16

9. A mass is supported on frictionless smooth horizontal surface. It is attached to a string rotated about a fixed centre at an angular velocity . If the length of the string and the angular velocity are doubled (the initial tension is T), then the tension in the string will be

a. T b. T/2

c. 4 T d. 8 T

10. A small coin is kept at the rim of a horizontal circular disc which is set into rotation about vertical axis passing through its centre. If radius of the disc is 5 cm and $\mu_s = 0.25$, then the angular speed at which the coin will just slip off at

a. 5 rad/s b. 7 rad/s

c. 10 rad/s d. 4.9 rad/s

11. The change in the centripetal force of a body moving in a circular path, if speed is made half and radius is made 4 times the original value, will

a. increase by b. decrease by

c. decrease by d. increase by

12. An object of mass 50 kg is moving in a horizontal circle of radius 8 m. If the centripetal force is 40 N, then the kinetic energy of an object will be

a. 320 J b. 260 J

c. 220 J d. 160 J

13. Two particles of equal masses are revolving in circular paths of radii 2 m and 8 m respectively with the same period. Then the ratio of their centripetal forces is,

a. 8 : 1 b. 2 : 1

c. 1 : 4 d. 1 : 1

14. A toy car weighing 1 kg tied at the end of a string 1 m long moves in a circle on the ground. What is the maximum possible speed of the car if the string has a breaking strength of 9 N?

a. 9 m/s b. 3 m/s
c. 1.5 m/s d. 0.75 s

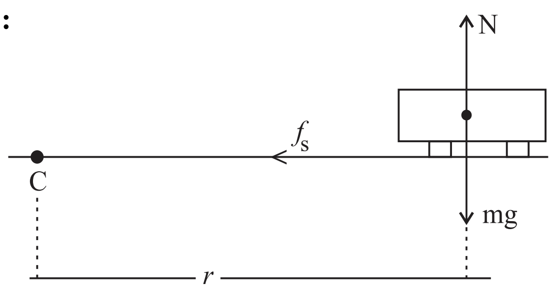
15. A string breaks if its tension exceeds 10 newtons. A stone of mass 250 gm tied string of length 10 cm is rotated in a horizontal circle. The maximum angular velocity of rotation can be

a. 20 rad/s b. 40 rad/s
c. 100 rad/s d. 200 rad/s

1.3 Application of Uniform Circular Motion

Q.4 Derive an expression for maximum safe speed of vehicle on a horizontal circular road.

Ans:



i. Consider a car of mass m taking turn on circular road of radius r.

Let μ_s be the coefficient of static friction between tyres and the surface of road.

ii. Consider car as a particle. Forces acting on car are

a. weight (mg), Vertically downward
b. normal reaction (N), Vertically upward
c. Force of static friction (fs) between road and tyres

iii. Normal reaction on car balances weight of car

$\therefore$ N = mg ... (1)

iv. The necessary centripetal force for circular motion of car is provided by frictional force between tyres and road which prevent car from outward skidding or slipping.

Let v_{max} be the maximum safe speed of car without skidding

$$\therefore \begin{pmatrix}\text{maximum}\\ \text{centripetal}\\ \text{force}\end{pmatrix} = \begin{pmatrix}\text{force due to}\\ \text{static friction}\end{pmatrix}$$

$$\frac{mv^2_{max}}{r} = \mu_s \cdot N$$

$$\therefore \frac{\cancel{m}v^2_{max}}{r} = \mu_s \cancel{m}g \qquad \text{...[from 1]}$$

$$v^2_{max} = \mu_s rg$$

$$\therefore \boxed{v_{max} = \sqrt{\mu_s rg}}$$

This is maximum speed of car on horizontal curve road.

Q.5 Define
i. Banking of road ii. Angle of banking

Ans. i. **Banking of Road :**
The arrangement of keeping the outer edge of the road surface inclined with the horizontal is called banking of road.

ii. **Angle of Banking :** The angle made by the inclined road surface with the horizontal is called angle of banking.

⋆Q.6 Why are curved roads banked ?

Ans.

i. When a vehicle is taking turn on horizontal curve road, the necessary centripetal force is provided by force of static friction between the tyres of the vehicle and road.

ii. Frictional force is non-reliable force. The value of frictional force is not constant as it depend upon the nature of the surfaces in contact and the presence of oil and water on the road. If the friction is inadequate, a speeding car may skid off the road.

iii. If we increase the friction by making surface of road rough it may result into fast wear and tear of the tyres.

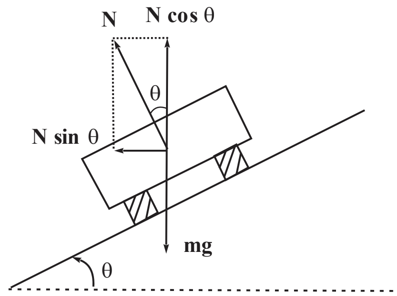
iv. To avoid the risk of skidding as well as to reduce the wear and tear of the car tyres, the outer side of road is raised above inner side. This is called banking of road.

v. When the road is banked, the horizontal component of normal reaction provides necessary centripetal force.

Q.7 Obtain expression for angle of banking when a vehicle moves along a curved banked road neglecting friction.

Ans.

i. The vertical section of vehicle on a curved road is shown in figure



Let r be the radius of curved road and θ be the angle of banking.

ii. Consider the vehicle to be a point and ignoring friction and other non-conservative forces.

iii. The two forces acting on vehicle are

a. weight (mg) acting vertically downward

b. normal reaction (N), perpendicular to the surface of road.

iv. Resolving N into two components

a. N sinθ - along the horizontal

b. N cosθ - along the vertical

v. The vertical component (N cosθ) balances weight (mg)

∴ $N\cos\theta = mg$ ...(1)

vi. Horizontal component (N sinθ) provides necessary centripetal force

∴ $N\sin\theta = \frac{mv^2}{r}$ ...(2)

vii. Dividing equation (2) by (1)

$$\frac{N\sin\theta}{N\cos\theta} = \frac{mv^2}{r} \div mg$$

$$\tan\theta = \frac{\cancel{m}v^2}{r} \times \frac{1}{\cancel{m}g}$$

$$\tan\theta = \frac{v^2}{rg}$$

$$\therefore \boxed{\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)}$$

This is expression for angle of banking when vehicle moves along a curve road neglecting friction.

Note:

Expression for most safe speed of vehicle on Banked road.

$$\tan\theta = \frac{v^2}{rg}$$

$$\therefore \quad v = \sqrt{rg\tan\theta}$$

Q.8 Obtain an expression for lower and upper speed limit for a vehicle moving on a banked road.

Ans :

i. Consider a vehicle on a rough curve road of radius r and banked at angle θ

ii. If a vehicle is running exactly at the speed $v_s = \sqrt{rg\tan\theta}$, the forces acting on the vehicle are

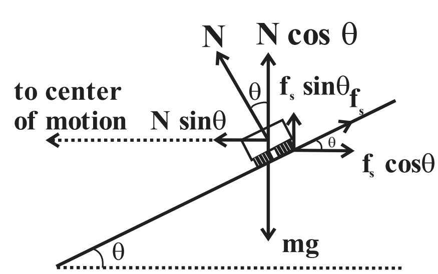
a. weight (mg) acting vertically downward.

b. normal reaction (N) acting perpendicular to the road.

iii. In practice, vehicle never travel exactly with this speed for speed other than this, the components of force of static friction between road and tyres help us upto a certain limit

iv. for speed $v_1 < \sqrt{rg\tan\theta}$,

$\frac{mv_1^2}{r} < N\sin\theta$ i.e. component $N\sin\theta$ is greater than the centrifugal force. In this case direction of force of static friction (f_s) between road and the tyres is directed along the inclination of the road, upward.



Resolving f_s into two perpendicular component.

a. $f_s \cos\theta$ is horizontal opposite to $N\sin\theta$ and

b. $f_s \sin\theta$ is vertically upward.

$N\sin\theta$ and $f_s\cos\theta$ take care of necessary centripetal force.

$$\therefore \quad \frac{mv_1^2}{r} = N\sin\theta - f_s\cos\theta$$

$$\therefore \quad \frac{mv_1^2}{r} = N\sin\theta - \mu_s N\cos\theta \qquad ...(1)$$

$(\because f_s = \mu_s N)$

The vertical component $N\cos\theta$ and $f_s\sin\theta$ balances weight mg

$$mg = N\cos\theta + f_s\sin\theta$$

$$mg = N\cos\theta + \mu_s N\sin\theta \qquad ...(2)$$

$(\because f_s = \mu_s N)$

Dividing (1) by (2)

$$\frac{mv_1^2}{r \times mg} = \frac{N\sin\theta - \mu_s N\cos\theta}{N\cos\theta + \mu_s N\sin\theta}$$

$$\frac{v_1^2}{rg} = \frac{N\cos\theta\left[\frac{\sin\theta}{\cos\theta} - \mu_s\right]}{N\cos\theta\left[1 + \mu_s\frac{\sin\theta}{\cos\theta}\right]}$$

$$\frac{v_1^2}{rg} = \left[\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right]$$

$$v_1^2 = rg\left[\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right]$$

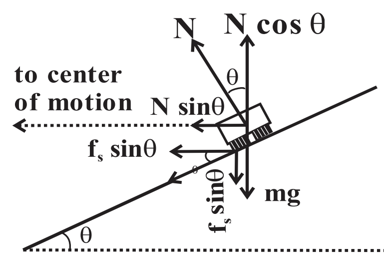
$$\therefore \quad \boxed{v_1 = \sqrt{rg\left[\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right]}}$$

for $\mu_s \geq \tan\theta$, $v_{min} = 0$. This is true for most of the rough roads, banked at smaller angles,

v. For speeds $v_2 > \sqrt{rg\tan\theta}$

The component of $N\sin\theta$ is less than the centrifugal force $\frac{mv_2^2}{r}$

$$\therefore \quad \frac{mv_2^2}{r} > N\sin\theta$$



In this case, the direction of force of static friction (f_s) between road and tyres is directed along the inclination of the road, downwards. Resolving f_s into two perpendicular components.

vi. Horizontal component ($f_s\cos\theta$) is parallel to $N\sin\theta$. These two forces take care of the necessary centrifugal force.

$$\frac{mv_2^2}{r} = N\sin\theta + f_s\cos\theta$$

$$\frac{mv_2^2}{r} = N\sin\theta + \mu_s N\cos\theta \qquad ...(3)$$

vii. Vertical component ($f_s\sin\theta$)is downward opposite to $N\cos\theta$

$$\therefore \quad mg = N\cos\theta - f_s\sin\theta$$

$$\therefore \quad mg = N\cos\theta - \mu_s N\sin\theta \qquad ...(4)$$

$(\therefore f_s = \mu_s N)$

Dividing (3) and (4)

$$\frac{mv_2^2}{r \times mg} = \frac{N\sin\theta + \mu_s N\cos\theta}{N\cos\theta - \mu_s N\sin\theta}$$

$$v_2^2 = rg\left[\frac{N\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \mu_s\right)}{N\cos\theta\left(1 - \mu_s\frac{\sin\theta}{\cos\theta}\right)}\right]$$

$$v_2^2 = rg\left[\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right]$$

$$\boxed{v_2 = \sqrt{rg\left[\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right]}}$$

If $\mu_s = \cot\theta$; $v_{max} = \infty$

But (μ_s)=1. Thus for $\theta \geq 45$

$v_{max} = \infty$. However for heavily banked road, minimum limit may be important.

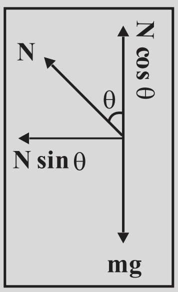
INTEXT QUESTION

Determine the angle to be made with the vertical by a two wheeler rider while turning on a horizontal track ?

Ans:

i. A cyclist negotiating a curve with certain speed has to lean through certain angle to the vertical to obtain the necessary centripetal force.

ii. When the cyclist bends at a curve , horizontal component normal reaction provides required centripetal force and vertical component balances the weight.



iii. Here $N \sin \theta = mv^2/r$ and

$N\cos \theta = mg$

$\tan \theta = v^2/rg$

Therefore, the rider should bend through an angle θ.

iv. In this case θ is independent of mass of the cyclist or cycle.

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

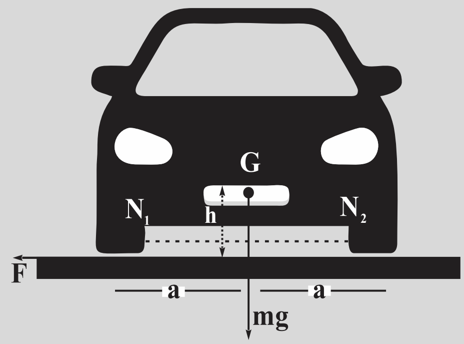
INTEXT QUESTION

Obtain the condition for not toppling for a four wheeler. On what factors does it depend, and in what way? Think about the normal reactions - Where are those and how much are those! What is the recommendation on loading the vehicel for not toppling easily? if a vehicles topples while turinig, which wheels leave the contact? Why? how does it affect the tyres? What is the recommendation for this?

Ans:

i. When a four wheeler takes a turn on a curve unbaked road then it has a tendency to skid away from centre of the road.The torque that prevents it from skidding away from the turn arises as the normal force on the outside wheels is larger than the normal force on the inside wheels.

ii. When a car moves in a circular path with speed more than a certain maximum speed, then it overturns even if friction is sufficient to avoid skidding and its inner wheel leaves the ground first.



iii. let m be the mass of car. Let car is moving with speed v on a curve road of radius r. Let 2a be the distance between the centres of the wheel of the car.

Let h be the height of the centre of gravity of car from road

Let N_1 be the reaction on inner wheel and N_2 be the reaction on outer wheel of the car.

iv. When car is taking turn on horizontal road necessary centripetal force is provided by frictional force

$$F = \frac{mv^2}{r} \quad ...(1)$$

v. When car is in rotational equilibrium net torque acting on car is zero. Taking moment of force about point G we get

$$Fh + N_1 a = N_2 a \quad ...(2)$$

vi. As ther is no vertical motion,

$$N_1 + N_2 = mg \quad ...(3)$$

Multiply equation (3) by a and add with (2)

$$\cancel{N_1a} + N_2a = Fh \quad ...(2)$$

$$\cancel{N_1a} + N_2a = mg\,a \quad ...(3)$$

$$2N_2a = Fh + mg\,a$$

$$2N_2 = \frac{Fh}{a} + mg$$

$$2N_2 = \frac{mv^2}{ra}h + mg$$

$$N_2 = \frac{m}{2}\left[\frac{v^2h}{ra} + g\right]$$

Similarly

$$N_1 = \frac{m}{2}\left[g - \frac{v^2h}{ra}\right]$$

$$N_1 = \frac{1}{2}m\left[g - \frac{v^2h}{ra}\right] \quad ...(4)$$

$$N_2 = \frac{1}{2}m\left[g + \frac{v^2h}{ra}\right] \quad ...(5)$$

vi. From equation (4), if v increases, value of N_1 decrease and for $N_1 = 0$,

$$\frac{v^2h}{ra} = g$$

$$v = \sqrt{\frac{gra}{h}}$$

i.e, the maximum speed of a car without overturning on a flat road is given by

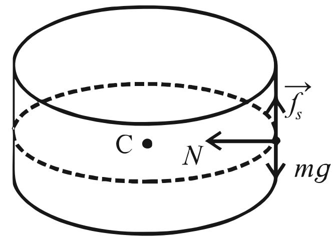
$$v = \sqrt{\frac{gra}{h}}$$

iv. as maximum speed is independent of mass of the vehicle does not play a vital role for toppling.

v. for safe driving of vehicle on curved unbanked road, the speed should be $v < \sqrt{\frac{gra}{h}}$ as friction is not reliable, at high speeds and sharp turns friction is not able to provide the required centripetal force. Friction causes unnecessary wear and tear of the tyres.

Q.9 Derive an expression for minimum safest velocity of a vehicle to move in well of death.

i. Consider a vertical cylindrical wall of radius r inside which a vehicle is driven in horizontal circles.



ii. Consider vehicle to be point object. Forces acting on the vehicle are

a. Normal reaction (N) acting horizontally and towards the centre.

b. Weight mg acting vertically downwards.

c. Force of static friction between vertical walls and the tyres. Force of static friction prevent the downward slipping.

iii. Acting upward static friction is equal to weight of vehicle.

$\therefore \quad mg = fs \quad ...(1)$

iv. For circular motion centripetal force is required that is provided by normal reaction on vehicle

$$\therefore \quad N = \frac{mv^2}{r} \quad ...(2)$$

v. Force of static friction is always less or equal to $\mu_s N$.

$\therefore \quad f_s \le \mu_s N$

$$\therefore \quad \cancel{m}g \le \mu_s\left(\frac{\cancel{m}v^2}{r}\right) \quad ...(\text{from 1 \& 2})$$

$$\therefore \quad g \le \frac{\mu_s v^2}{r}$$

$$\therefore \quad v^2 \ge \frac{rg}{\mu_s}$$

$$\therefore \quad \boxed{v_{min} = \sqrt{\frac{rg}{\mu s}}}$$

This is safest minimum velocity of vehicle to move in well of death.

Type - IV

Numerical based on speed of vehicle on road

Formulae used

1. velocity of car on horizontal road

$$V_{max} = \sqrt{\mu_s rg}$$

2. velocity of car on banked road

$$V = \sqrt{rg \tan\theta}$$

3. Angle of banking

$$\tan\theta = \frac{v^2}{rg} \qquad \theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

4. Maximum speed limit

$$v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta}\right)}$$

5. Minimum speed limit

$$v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta}\right)}$$

1) A motor van weighing 4400 kg rounds a level curve of radius 100 m on unbanked road at 60 km/hr. What should be minimum value of coefficient of friction to prevent skidding ? At what angle the road should be banked for this velocity.

Data : m = 4400 kg., r = 100 m,

$$v = 60 \text{ km/hr} = \overset{10}{\cancel{60}} \times \frac{5}{\underset{3}{\cancel{18}}} = \frac{50}{3} \text{ m/s}$$

To find: i. μ ii. θ

Formula: i. To prevent skidding

$$\frac{mv^2}{r} = \mu mg$$

$$\therefore \quad \mu = \frac{v^2}{rg}$$

ii. $\tan\theta = \frac{v^2}{rg}$

Solution :

$$\mu = \frac{v^2}{rg} = \frac{(50/3)^2}{100\times9.8} = \frac{2500}{100\times9.8\times9}$$

$$\mu = 0.2834$$

ii. $\tan\theta = \frac{v^2}{rg} = 0.2834$

$$\theta = \tan^{-1}(0.2834)$$

$$\theta = 15°49'$$

Ans : i. The coefficient of friction is 0.2834
ii. Angle of banking is 15° 49'

★2) A racing track of curvature 9.9 m is banked at $\tan^{-1}(0.5)$. Coefficient of static friction between the track and the tyres of a vehicle is 0.2. Determine the speed limits with 10% margin. (Take g = 10 m/s²)

Data: r = 9.9 m, $\mu_s = 0.2$,
$\theta = \tan^{-1}(0.5)$ $\tan\theta = 0.5$

To find: Maximum and minimum speed limit(under 10% margin)

Formulae: i. $v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta}\right)}$

ii. $v_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta}\right)}$

Solution:

i For minimum speed limit

$$v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s \tan\theta}\right)}$$

$$v_{min} = \sqrt{9.9\times10\left(\frac{0.5-0.2}{1+(0.2\times0.5)}\right)}$$

$$= \sqrt{99\times\frac{0.3}{1.1}} = \sqrt{27} = 5.196 \text{ m/s}$$

To keep 10 % margin, allowed v_{min} should be 10% higher than this,

$$\therefore \quad (v_{min})_{allowed} = 5.196\times\frac{110}{100} = 5.716 \text{ m/s}$$

ii. $V_{max} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s \tan\theta}\right)}$

$$v_{max} = \sqrt{9.9 \times 10\left(\frac{0.5+0.2}{1-(0.2\times 0.5)}\right)}$$

$$= \sqrt{99\left(\frac{0.7}{0.9}\right)} = \sqrt{77} = 8.775 \text{ m/s}$$

To keep 10 % margin, allowed v_{max} should be 10% lower than this,

$$\therefore \quad (v_{min})_{allowed} = 8.775 \times \frac{90}{100} = 7.898 \text{ m/s}$$

> **Ans :** i. The minimum velocity allowed is 5.716 m / s.
> ii. The maximum velocity allowed is 7.898 m / s.

3) The road in the Q.2 is constructed as per the requirements. The coefficient of static friction between the tyres of a vehicle on this road is 0.8, will there be any lower speed limit? By how much can the upper speed limit exceed in this case?

Data: $\tan\theta = 5$, $\mu_s = 0.8$, r = 72 m

To find: i. v_{min} ii. v_{max}

Formula: Minimum speed limit

$$V_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1+\mu_s\tan\theta}\right)}$$

Solution :

i. Minimum speed limit of banked road

$$V_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1+\mu_s\tan\theta}\right)}$$

$$v = \sqrt{72\times 10\left(\frac{5-0.8}{1+0.8\times 5}\right)}$$

$$= \sqrt{144\times 4.2} = \sqrt{604.8} = 24.58 \text{ m/s}$$

= 88.52 km/h

ii. For maximum speed limit

we know that, $\theta = 78.69°$

Since the track is heavily banked, $\theta \geq 45°$, there is no upper limit on this track

> **Ans :** Minimum speed limit is 88.52 km/hr and no upper limit

★4) Part of a racing track is to be designed for a curvature of 72 m. We are not recommending the vehicles to drive faster than 216 km/hr. With what angle should the road be tilted? By what height will its outer edge be, with respect to the inner edge if the track is 10 m wide? [Take g = 10 m/s²]

Data: r = 72m,

$$v_{max} = 216 \text{ kmph} = 216 \times \frac{5}{18} = 60 \text{ m/s}$$

Width of track (l) = 10 m

To Find: i. Angle of banking (θ)

ii. Elevation of outer edge over inner (h)

Formula: $\tan\theta = \frac{v^2}{rg}$

Solution:

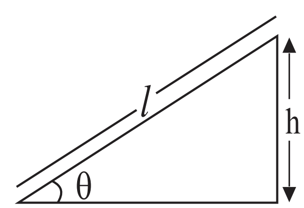
i. $\tan\theta = \frac{v^2}{rg}$

$$\tan\theta = \frac{(60)^2}{72\times 10}$$

$\tan\theta = 5$

$\theta = \tan^{-1}(5) = 78°42'$

ii.



$h = 10\sin\theta$

$= 10\sin = 78°42'$

$= 9.806$ m

> **Ans :** i. The angle by which road should be titled is 78°42'
> ii. The height of its outer edge w.r.t. its inner edge is 9.806 m.

Problem for Practice

1. A flat curve on a highway has radius of curvature 400 m. A car goes around a curve at a speed of 32 m/s. What is the minimum value of coefficient of friction that will prevent car from sliding? **Ans : 0.261**

2. A vehicle is moving on a circular track whose surface is inclined towards the horizontal at an angle of 10°. The maximum velocity with which it can move safely is 36 km/hr. Calculate length of circular track.

Ans : 363.72m

3. A vehicle moves along a circular road which is inclined to the horizontal at 10^0. The maximum velocity with which it can move safely is 36 km/h. Calculate the length of the circular road.

Ans : 363.5 m

4. The radius of a circular curve on a road is 300 m. The road is banked at an angle of 15^0. Find the optimum speed of a vehicle which will avoid wear and tear on its tyres. Find also the maximum speed find also the maximum speed for safe driving if μ=0.2

Ans : 28 m/s; 38 m/s

5. A circular race track of radius 400 m is banked at angle of 100. If the coefficient of friction between the wheel of a race car and the road is 0.2 what is the (1) optimum speed of the race car to avoid wear and tear on its tyres (2) maximum permissible speed to avoid slipping ?

Ans: 26.29 m/s, 39.10 m/s

6. A vehicle enters a circular bend of radius 200 m at 72 km / h. The road surface at the bend is banked at 10^0. Is it s afe ? At what angle should the road surface be ideally banked for safe driving at this speed? If the road is 5 m wide, what should be the elevation of the outer edge of road surface above the inner edge?

Ans : Safety speed limit
66.9 km/h; $11^0$53'; 1m

7. A bend level in road has a radius 100m. Find the maximum speed which car turning this bend may have without skidding, if coefficient of friction between the tyre and road is 0.8

Ans: 28 m/s

Type - V
Numerical based on angle of inclination of motorcyclist

Formulae used

Angle made by motorcyclis + rider with vertical

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

1) Find the angle which the bicycle and its rider make with the vertical when going at 18 km/h around a curved road of radius 10m on level ground g = 9.8 m/s².

Data : $v = 18\frac{km}{hr} = 18 \times \frac{5}{18} = 5 m/s$,r = 10 m

To find : Angle made by the bicycle and its rider make with the vertical (θ)

Formula: $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$

Solution: $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$

$$\theta = \tan^{-1}\left(\frac{5^2}{10 \times 9.8}\right) = \tan^{-1}(0.2551)$$

$\therefore \quad \theta = 14^0\ 18'$

Ans : Angle made by the bicycle and its rider make with the vertical is 14° 18'

2) A motor cyclist rounds a curve of radius 25 m at 36 km/hr . The combined mass of motorcycle and man is 150 kg

i) What is centripetal force exerted on motor cyclist ?

ii) What is upward force exerted on motor cyclist ?

iii) What angle the bicycle makes with vertical?

Data : m = 150 kg, r = 25 m,

$$v = 36\ km/h = 36 \times \frac{5}{18} = 10\ m/s$$

To Find : i. F ii. N iii. θ

Formula :

i. $F = \frac{mv^2}{r}$

ii. $N = mg$

iii. $\tan\theta = \frac{v^2}{rg}$

Solution:

i. $F = \frac{mv^2}{r} = \frac{\overset{6}{\cancel{150}} \times (10)^2}{\cancel{25}} = 600\text{ N}$

ii. $N = mg = 150 \times 9.8 = 1470\text{ N}$

iii. $\tan\theta = \frac{v^2}{rg} = \frac{(10)^2}{25 \times 9.8} = 0.4083$

∴ $\theta = \tan^{-1}(0.4083) = 22^0 12'$

Ans :	i. Centripetal force is 600 N ii. Normal reaction is 1470 N iii. Angle made with verticle is $22^0 12'$

Problem for Practice

1. Find the angle which the bicycle and its rider make with the vertical when going at 27 km/h around a curve of 10 m radius .

Ans : $29^0 51'$

2. A motor cyclist goes along a circular curve at 120 km/h. How far from the vertical must he lean inwards for balance if the track is 3 km long? $g = 9.8\text{ m/s}^2$.

Ans : $13^0 20'$

Type - VI
Numerical based on banking of aircraft

Formulae

Angle of banking of Aircraft

$\tan\theta = \frac{v^2}{rg}$

1) **An aircraft in level flight completes a circular turn in 100 second.**

i. What is the radius of circular turn ?

ii. What is the angle of banking, if the velocity of aircraft is 40 m/s ?

Data T = 100 s, v = 40 m/s

To find: i. r ii. θ

Formulae i. $\omega = \frac{2\pi}{T}$ ii. $v = r\omega$ iii. $\tan\theta = \frac{v^2}{rg}$

Solution

i. $\omega = \frac{2\pi}{T} = \frac{2\pi}{100} = \frac{\pi}{50}\text{ rad/s}$

ii. $v = r\omega$

∴ $r = \frac{v}{\omega} = \frac{40 \times 50}{\pi} = \frac{2000}{3.14} = 636.5\text{m}$

iii. $\tan\theta = \frac{v^2}{rg} = \frac{(40)^2}{(636.5) \times 9.8} = 0.2565$

∴ $\theta = \tan^{-1}(0.2565) = 14^0 25'$

Ans :	Radius of circular track is 636.5 and angle in banking is $14^0 25'$

Problem for Practice

1. An aircraft takes a turn along a circular path of radius 1500m, if the linear speed of the aircraft is 300 m/s. Find its angular speed and time taken by it to complete $(1/5)^{th}$ of the circular path.

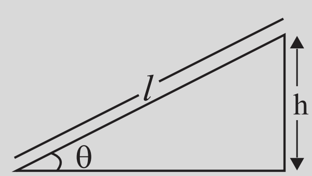
Ans : 0.2 rad/s, 6.28 sec

Type - VII
Numerical based on Banking of train

Formulae used

1. Angle of banking $\tan\theta = \frac{v^2}{rg}$

2. Elevation of outer rail over inner rail (h)



$h = l \sin\theta$

1) **A meter gauge train is moving at 60 km/hr along a curved road of radius of curvature 500 m at a certain place. Find the elevation of outer rail above inner rail, so that there is no side pressure on rail ($g = 9.8\text{ m/s}^2$)**

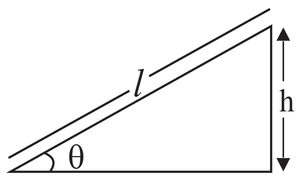
Data: $v = 60 \text{ km/hr} = 60 \times 5 = \frac{50}{3}$ m/s

$l = 1$ m, r = 500 m

To find : Elevation of outer rail over inner rail (h)

formulae : i. $\tan\theta = \frac{v^2}{rg}$

ii.



$h = l \sin\theta$

Solution:

i. $$\tan\theta = \frac{v^2}{rg} = \frac{\left(\frac{50}{3}\right)^2}{500 \times 9.8}$$

$\tan\theta = 0.05670$

∴ $\theta = \tan^{-1}(0.05670)$

∴ $\theta = 3^{\circ}15'$

ii. Elevation of outer rail above inner rail,

$h = l \sin\theta$

$h = (1) \sin 3^{\circ}15'$

$h = 0.0567$ m

Ans : Elevation of outer rail over inner rail is 0.0567 m

Problem for Practice

1. Find the angle of banking of the railway track of radius of curvature 1600 meters. If the optimum velocity of the train is 20 m/s. Also find the elevation of the outer track above the inner track if the distance between the two tracks is 1.8 m.

Ans: 1^0 28', 0.0459 m

2. Find the angle of banking of the railway track of radius of curvature 3200 m if there is no side thrust on the rails for a train running at 144 km/h. Find the elevation of the outer rail above the inner one if the distance between the rails is 1.6m.

Ans : $2^0$55';81 mm

3. A railway track goes round a curve having a radius of curvature 1 km.The distance between the rails is 1 m. Find the elevation of the outer rail above the inner rail so that there is no side pressure against the rails when a train goes round the curve at 36 km/hr.

Ans: 1.02 cm

Type - VIII
Numerical based on death well

Formulae used

$$v = \sqrt{\frac{rg}{\mu_s}}$$

★1) A motor cyclist (to be treated as a point mass) is to undertake horizontal circles inside the cylindrical wall of a well of inner radius 4 m. Coefficient of static friction between the tyres and the wall is 0.4. Calculatethe minimum speed and frequency necessary to perform this stunt. (Use g = 10 m,/s²)

Data: r = 4 m, $\mu_s = 0.4$, g = 10 m/s²

To Find: i. Minimum speed to perform stunt (v_{min})

ii. Minimum frequency of perform stunt(n_{min})

Formulae: i. $v = \sqrt{\frac{rg}{\mu_s}}$

ii. $v = 2\pi nr$

Solution:

i. $$v = \sqrt{\frac{rg}{\mu_s}}$$

$$v_{min} = \sqrt{\frac{4 \times 10}{0.4}} = 10 \text{ ms}^{-1}$$

ii. $v = 2\pi nr$

$v_{min} = 2\pi n_{min} r$

$$n_{min} = \frac{v_{min}}{2\pi r} = \frac{10}{2 \times \pi \times 4} = \frac{5}{4 \times 3.142}$$

= 0.4 rev/s

Ans : i. Minimum speed required to perform stunt is $10\,ms^{-1}$
ii. Minimum frequency required to perform stunt is $0.4\,rev\,s^{-1}$

★ 2) During a stunt, a cyclist (Consider to be a particle) is undertaking horizontal circles inside a cylindrical well of radius 6.05m. If the necessary friction coefficient is 0.5, how much minimum speed should the stunt artist maintain? Mass of the artist is 50 kg. If She/he increases the speed by 20%, how much will the force of friction be?

Data: $r = 6.05$ m, $\mu_s = 0.5$, $M = 50$ kg,
For case B, $v_2 = v_1 + 20\%$ of $v_1 = 1.2v_1$

To find : i. Minimum speed maintained by stunt-artist (v_{min})
ii. Force of friction if velocity is increased to 20%

Formulae: i. $v = \sqrt{2rg}$
ii. $F_s = mg$

Solution:

i. $v = \sqrt{2rg} = \sqrt{2 \times 6.05 \times 10} = \sqrt{121} = 11\,m/s$

ii. Force of friction,
$F_s = mg = 50 \times 10 = 500N$

Ans : i. The minimum speed maintained by stunt artist (v_{min}) is $11\,m/s$.
ii. The force of friction is 500 N

Problem for Practice

1. A person wants to drive on the vertical surface of a large cylindrical wooden 'well' commonly knows as 'death well' in a circus. The radius of the well is R and the coefficient of friction between the tyres of the motorcycle and the wall of the well is μ_s. The minimum speed the motor cyclist must have in order to prevent slipping should be

Ans : $\sqrt{\frac{Rg}{\mu_s}}$

2. A person wants to drive on the vertical surface of a large cylindrical wooden well commonly known as death well in a circus. The radius of well is 2 meter and the coefficient of friction between the tyres of the motorcycle and the wall of the well is 0.2, the minimum speed the motorcyclist must have in order to prevent slipping should be

Ans : 1.98m/s

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 4)

1. A cyclist turns around a curve at 15 miles/hour. If he turns at double the speed, the tendency to over turn is
a. Doubled
b. Quadrupled
c. Halved
d. Unchanged

2. A cyclist taking turn bends inwards while a car passenger taking same turn is thrown outwards. The reason is
a. Car is heavier than cycle
b. Car has four wheels while cycle has 02 only two
c. Difference in the speed of the two
d. Cyclist has to counteract the centrifugal force while in the case of car only the passenger is thrown by this force

3. A car sometimes overturns while taking a turn.When it overturns, it is
a. The inner wheel which leaves the ground first
b. The outer wheel which leaves the ground first
c. Both the wheels leave the ground simultaneously
d. Either wheel leaves the ground first

4. A circular road of radius 1000 m has banking angle 45°. The maximum safe speed of a car having mass 2000 *kg* will be, if the coefficient of friction between tyre and road is 0.5
a. 172 m/s
b. 124 m/s
c. 99 m/s
d. 86 m/s

5. If a cyclist moving with a speed of 4.9 *m/s* on a level road can take a sharp circular turn of radius 4 m, then coefficient of friction between the cycle tyres and road is

a. 0.41 b. 0.51
c. 0.61 d. 0.71

6. A person is in contact with the inner wall of a vertical hollow cylinder of radius 1m, remains in equilibrium without slipping down as the cylinder is rotated about its own vertical axis with an angular velocity 3.5 rad/s.The minimum coefficient of static friction between person and wall of cylinder such that the person does not slip down is

a. 0.2 b. 0.4
c. 0.6 d. 0.8

7. A person is in contact with inner wall of a vertical hollow cylinder of radius 2 m without floor under his feet, remains in equilibrium without slipping down when the cylinder is rotated about it's own vertical axis. If the coefficient of static friction between person and wall of cylinder is 0.4, the minimum angular velocity of cylinder should be

a. 1.5 rad/s b. 2 rad/s
c. 3 rad/s d. 3.5 rad/s

8. A coin placed on a rotating turn table just slips if it is placed at a distance of 20 cm from the centre. If the angular velocity of the turn table is doubled, the coin will just slip at a distance of

a. 40 cm b. 20 cm
c. 10 cm c. 5 cm

9. A cyclist goes round a circular path of circumference 34.3 m in $\sqrt{22}$ s. The angle made by him, with the vertical, is

a. 42° b. 43°
c. 44° d. 45°

10. A train is moving at 20 m/s on a railway track of radius of curvature 1600 m. If the distance between two tracks is 1.8 m, then the elevation of the outer track above the inner track will be (g = 10 m/s^2)

a. 0.450 m b. 0.0450 m
c. 4.50 m d. 4.0 m

Try Yourself

11. A motor cyclist rides along a horizontal circle on the vertical cylindrical wall of a metal cylinder. The radius of the cylinder is 10 m. If the speed is 20 m/s and acceleration due to gravity is 10 m/s^2, then the least value of the coefficient of friction will be,

a. 0.25 b. 0.45
c. 0.35 d. 0.15

12. A cyclist turns around a curve at 50 km/h. If it rounds the curve to double the speed, its tendency to overturn would be,

a. halved b. tripled
c. doubled d. four times

13. A vehicle moves on a horizontal curved road of radius of curvature 50 m, height of centre of gravity 1.5 m, the distance between the two wheels 2 m and acceleration due to gravity 9.8 m/s^2. Then the maximum velocity with which it can travel on the road will be

a. 18 m/s b. 20 m/s
c. 19 m/s d. 17 m/s

14. The height of the centre of gravity of the truck above the ground is 1.5 m and the distance between the wheels is 1.5 m. If the maximum velocity at which a truck can safely travel along the horizontal track without toppling on a curve of radius 250 m will be

a. 30 m/s b. 35 m/s
c. 40 m/s d. 45 m/s

15. The radius of curvature of a metre gauge railway line at a place where the train is moving at 36 km/h is 50 m. If there is no side thrust on the rails, then the elevation of the outer rail above the inner rail will be (g = 10 m/s^2)

a. 5 m b. 2 m
c. 0.5 m d. 0.2 m

16. A cyclist is riding with a speed of 27 km/h. As he approaches a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at constant rate of 0.5 m/s^2. The magnitude of the net acceleration of the cyclist is

a. 0.86 m/s^2 b. 0.43 m/s^2
c. 1.24 m/s^2 d. 1.76 m/s^2

17. A car is moving with a speed of 10 m/s in a concave road of radius 100 m. If the mass of the car is 700 kg, then the reaction on the car tyres when it is at the lowest position will be

a. 4560 N b. 5560 N
c. 6560 N d. 7560 N

18. A road is banked with an angle 0.01 radian. If the radius of the road is 80 m ($g = 10\ m/s^2$) then the safe velocity for the drive will be

a. 4.8 m/s b. 2.8 m/s
c. 3.8 m/s d. 5.8 m/s

19. The maximum safe speed of a vehicle over a curved road of radius 150 m is 10 m/s. If the width of road is 7.5 m, the height of the outer edge will be

a. 0.25 m b. 0.50 m
c. 0.35 m d. 0.60 m

20. A curved road of 50 m radius is banked at correct angle for a given speed. If the speed is to be doubled keeping the same banking angle, the radius of curvature of road should be changed-to

a. 25 m b. 100 m c. 200 m d. 400 m

Conical Pendulum

Q.10 Define conical pendulum. Derive an expression for

i. speed of the bob

ii. angular velocity (ω)

iii. time period (T)

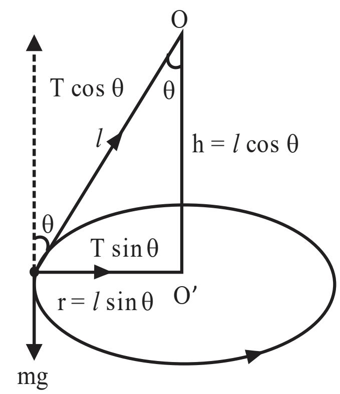
OR

★ **Show that time period of conical pendulum is**

$$T = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

Ans: **Definition :** Conical pendulum is a simple pendulum, which is given such a motion that bob describes a horizontal circle and the string describes a cone.

T = tension in the string
l = length of the string ,
h = axial height of the cone ,
v = velocity,
r = radius of horizontal circle ,
θ = semi vertical angle of cone ,
mg = weight of the sphere (bob)



Consider a bob of mass m revolving along a horizontal circle of radius r with velocity v. Let θ be semi vertical angle of cone. The forces acting on bob at position are,

i. Weight mg acting vertically downward.

ii. Tension T in upwards along the string. Tension (T) in the string can be resolved into

a. T cos θ vertically upward force balances weight.

b. T sin θ horizontal force and directed towards the centre of circle balances necessary centripetal force.

1. Linear speed :

$$T\cos\theta = mg \quad ...(i)$$

$$T\sin\theta = \frac{mv^2}{r} \quad ...(ii)$$

Dividing (ii) by (i), we get,

$$\tan\theta = \frac{v^2}{rg}$$

$$\therefore \quad v = \sqrt{r g \tan\theta}$$

= speed of the bob of conical pendulum.

ii. Angular speed :

$$v = r\omega$$

$\therefore \quad r\,\omega = \sqrt{r\,g\tan\theta} \qquad (\because\ v = \sqrt{r\,g\tan\theta}\ )$

since, $\tan\theta = \frac{r}{h}$ (from diagram)

$\therefore \quad r^2\,\omega^2 = r\,g\left(\frac{r}{h}\right)$

$\omega^2 = \frac{g}{h}$

$\therefore \quad \omega = \sqrt{\frac{g}{h}}$

= angular speed of the bob of conical pendulum

iii. Periodic time :

$T = \frac{2\pi r}{v}$

$T = \frac{2\pi r}{\sqrt{r\,g\tan\theta}}$

$T = 2\pi\sqrt{\frac{r}{g\tan\theta}}$ ... (iii)

since, $\sin\theta = \frac{r}{\ell}$ (from diagram)

$\therefore \quad r = \ell\sin\theta$ substituting the value in eqn. (iii),

$T = 2\pi\sqrt{\frac{\ell\sin\theta}{g\tan\theta}}$

$T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$ ... (iv)

Eqns. (iii) and (iv) give expressions for period of conical pendulum.

If $\angle\theta$ is small, $\cos\theta \cong 1$

$\therefore \quad T = 2\pi\sqrt{\frac{\ell}{g}}$

Thus, time period is independent of θ if ∠θ is small.

The period of conical pendulum depends on

i. length of pendulum
ii. angle of inclination to vertical
iii. acceleration due to gravity at a given place
iv. is independent of mass of bob.
v. independent of ∠θ (if θ is small).

★Q.11 On what vectors does the frequency of conical pendulum depend? is it independent of some factors?

Ans: The frequency of a conical of a pendulum, of string length l and semivertical angle θ, is

$$n = \frac{1}{2\pi}\sqrt{\frac{g}{l\cos\theta}}$$

where g is the acceleration due to gravity at the place. From the above expression, we can see that

i. $n \,\alpha\, \sqrt{g}$

ii. $n \,\alpha\, \frac{1}{\sqrt{l}}$

iii. $n \,\alpha\, \frac{1}{\sqrt{\cos\theta}}$

(if θ increases, cos θ decreases and n increases)

iv. The frequency is independent of the mass of the bob.

Type - IX
Numerical based on conical pendulum

Formulae used

$T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$

★1) A pendulum consisting of a massless string of length 20 cm and a tiny bob of mass 100 g is set up as a conical pendulum. Its bob now performs 75 rpm. Calculate kinetic energy and increase in the gravitational potential energy of the bob.

(Use $\pi^2 = 10$, $\cos\theta = 0.8$)

Data: l = 20cm =0.2m, m = 100g, = 0.1 kg,

$n = 75\text{ rpm} = \frac{75}{60} = \frac{5}{4}\text{Hz}$

$\omega = 2\pi n = 2\pi\times\frac{5}{4} = \frac{5}{2}\pi$ rad/s

To find: i. Kinetic energy of the bob (K.E)

ii. Increase in gravitational potential energy of the bob w.r.t. inner edge (Δ P.E)

Formulae: i. $T = \frac{1}{n}$

ii. $T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$

iii. $K.E. = \frac{1}{2}mv^2$

iv. $\Delta P.E. = mg(\ell - h)$

v. Increase in P.E.

$\Delta P.E$ = initial P.E. - final P.E.
= mg$l$ - mgh
= mg(l - h)

Solution:

i. $T = \frac{1}{n} = \frac{4}{5} = 0.8\,sec$

ii. $T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$

$\therefore$ $T^2 = 4\pi^2\frac{l\cos\theta}{g}$

$l\cos\theta = \frac{gT^2}{4\pi^2} = \frac{10\times(0.8)^2}{4\times10}$

$l\cos\theta = \frac{0.64}{4} = 0.16$

$\therefore$ $l\cos\theta = 0.16$

$\cos\theta = \frac{0.16}{l} = \frac{0.16}{0.2} = 0.8$

iii. $\theta = \cos^{-1}(0.8) = 36°5'$

$r = l\sin\theta$

r = 0.2 sin 36°5'

= 0.2 × 0.6 = 0.12

iv. $K.E. = \frac{1}{2}mv^2$

$= \frac{1}{2}mr^2\omega^2$ $(\because v = rw)$

$= \frac{1}{2}\times0.1\times(0.12)^2\times\left(\frac{5}{2}\pi\right)^2 = 0.045\ J$

v. Increase in P.E.

$\Delta P.E$ = initial P.E. - final P.E.
= mgℓ - mgh
= mg(ℓ - h)
= 0.1 × 10 [0.2 - 0.16] = 0.04 J

2) A string of length 0.5 m carries a bob of mass 0.1 kg at its end. It is used as a conical pendulum with a period 1.41sec. Calculate angle of inclination of string with vertical and tension in the string.

Data: $l = 0.5m,\ m = 0.1kg,\ T = 1.41\ sec$

To Find: i. θ ii. T'

Formula :

i. $T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$

ii. If tension in the string is T' then

$T'\cos\theta = mg$

$\therefore$ $T' = \frac{mg}{\cos\theta}$

Solution :

i. $T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}}$

$\therefore$ $1.41 = 2\times3.142\sqrt{\frac{0.5\cos\theta}{9.8}}$

$\therefore$ $\frac{1.41}{2\times3.142} = \sqrt{\frac{\cos\theta}{19.6}}$

$\therefore$ $\cos\theta = 19.6\left(\frac{1.41}{2\times3.142}\right)^2$

$\therefore$ $\cos\theta = 0.9947$

$\therefore$ $\theta = \cos^{-1}(0.9947)$

$\therefore$ $\theta = 9^0 5'$

ii. $T' = \frac{mg}{\cos\theta} = \frac{0.1\times9.8}{\cos 9^0 5'} = \frac{0.98}{0.9947} = 0.9858\ N$

Ans :	Angle made with verticle is $9^0 5'$ and tension in string is 0.9858 N

Problem for Practice

1. A stone of mass 1 kg is whirled in a horizontal circle attached at the end of 1 m long string. If the string makes an angle of 30^0 with vertical calculate.
   i. period ii. centripetal force($g = 9.8\ m/s^2$)

   Ans : 1.868 s, 5.658 N

2. A conical pendulum has a length of 1.5m and a bob of mass 50 g. The bob completes 20 revolutions in 45 s. Find the radius of the circular path traced by the bob and the tension in the thread. g= 9.8 m/s^2.

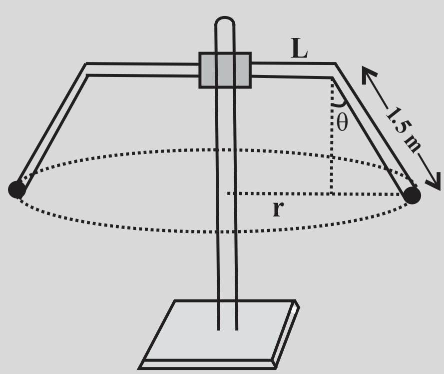
   Ans : 186 m / s

3. In a conical pendulum, a string of length 120 cm is fixed at rigid support and carries a mass of 150 g at its free end. If the mass is revolved in a horizontal circle of radius 0.2 m around a vertical axis calculate tension in the string ($g = 9.8\ m/s^2$)

Ans : 1.52 N

INTEXT NUMERICAL

A merry - go - round usually consists of a central vertical pillar. At the top of it there are horizontal rods which can rotate about vertical axis. At the end of this horizontal rod there is a verical rod fitted like an elbow joint. At the lower end of each vertical rod, there is a horse on which the rider can sit. As the merry - go round is set into rotation, these vertical rods move away from the axle by making some angle with the vertical.



The figure above shows vertical section of a merry-go-round in which the initially vertical rods are inclined with the vertical at 37° during rotation. Calculate the frequency of revolution of the merry-go-round.
(Use $g = \pi^2\ m/s^2$ and $\sin 37° = 0.6$)

Solution :

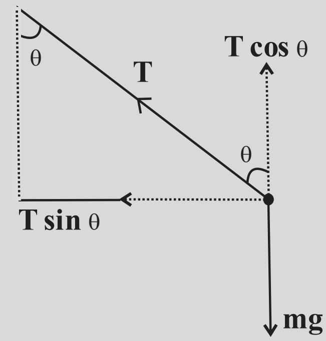
i. Length of the horizontal rod, H = 2.1m
Length of the 'initially vertical' rod, v = 1.5m
$\theta = 37°$

ii. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\therefore \quad \tan\theta = \dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \dfrac{0.6}{\sqrt{1-0.6^2}} = \dfrac{3}{4}$

iii. Radius of the horizontal circular motion of the rider,
$r = H + v\sin 37°$
$= 2.1 + (1.5 \times 0.6) = 3.0$ m

iv. If T is the tension along the inclined rod.



$T\cos\theta = mg$ ...(1)

$T\sin\theta = mr\omega^2$ ...(2)

Dividing equation (2) by (1), we get

$\tan\theta = \dfrac{\omega^2 r}{g} = \dfrac{4\pi^2 n^2 r}{g}$ ...($\because \omega = 2\pi n$)

$\tan\theta = 4n^2 r$ ...($\because g = \pi^2$)

$\therefore \quad n = \sqrt{\dfrac{\tan\theta}{4r}} = \sqrt{\dfrac{3}{4\times4\times3}} = 0.25$ rev s^{-1}

Ans : The frequency of revolution of the merry -go - round is 0.25 rev s -1.

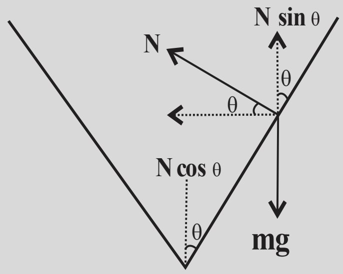
INTEXT NUMERICAL

Simi-vertical angle of the vonical section of a funnel is 37°. There is a small ball kept inside the funnel. On rotating the funnel, the maximum speed that the funnel is 2 m/s. Calculate inner radius of the rim of the funnel. Is there any limit upon the frequency of rotation? How much is it? Is it lower or upper limit? Give a logical reasoning.

(Use $g = 10\ m/s^2$ and $\sin 37° = 0.6$)

Solution:

$\tan\theta = \dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \dfrac{0.6}{\sqrt{1-0.6^2}} = \dfrac{3}{4}$



$N \sin\theta = mg$ ...(1)

$N \cos\theta = \frac{mv^2}{r}$ ...(2)

Dividing equiation (i) by (ii),

$\therefore \quad \tan\theta = \frac{rg}{v^2}$

$\therefore \quad r = \frac{v^2 \tan\theta}{g}$

$\therefore \quad r_{max} = \frac{v^2_{max} \tan\theta}{g} = \frac{2^2 \times 3}{4 \times 10} = 0.3m$

Now, $v = r\omega = 2\pi r n$

For the lower limit of the speed (while rotating), v approaches 0. This implies that $r \to 0$, but the frequency n increases. Hence a specific upper limit is not possible in the case of frequency. Thus, the practical limit on the frequencyof rotation is its lower limit. It will be possible for $r = r_{max}$

$\therefore \quad n_{max} = \frac{v_{max}}{2\pi r_{max}} = \frac{2}{2\pi \times 0.3} = \frac{1}{0.3\pi} \approx 1 \text{ rev/s}$

Ans : The inner radius of rim is 0.3 m and the minimum frequency of rotaion is 1 rev / s.

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 5)

1. A car is moving along a circular track of radius $10\sqrt{3}$ m with a constant speed of 36 kmph. A plumb bob is suspended from the roof of the car by a light rigid rod of length 1m. The angle made by the rod with the track is ($g = 10ms^{-2}$)

a. Zero b. 30°
c. 45° d. 60°

2. A particle describes a horizontal circle on the smooth inner surface of conical funnel whose vertex angle is 90°. If the height of the plane of the circle above the vertex is 9.8 cm, the speed of the particle is

a. $\sqrt{9.8}$ m / sec b. 0.98 m/sec
c. 19.6 m/sec c. 14.7 m/sec

3. In a conical pendulum, when the bob moves in a horizontal circle of radius r, with uniform speed v, the string of length L describes a cone of semi vertical angle θ. The tension in the string is given by

a. $T = \frac{mgL}{(L^2 - r^2)}$ b. $\frac{(L^2 - r^2)^{1/2}}{mgL}$

c. $T = \frac{mgL}{\sqrt{L^2 - r^2}}$ d. $T = \frac{mgL}{(L^2 - r^2)^2}$

4. A block of mass 10 kg is in contant against the inner wall of a hollow cylindrical drum of radius 4 m. The coefficient of friction between the block and the inner wall of the cylinder is 0.1. The minimum angular velocity needed for the cylinder to keep the block stationary when the cylinder is vertical and rotating about its axis, will be : ($g = 10 \text{ m/s}^2$)

a. 10 rad/s b. $\frac{10}{2\pi}$ rad/s

c. $\frac{5}{2\pi}$ rad/s d. 5 rad/s

Try Yourself

5. A car is moving in a horizontal circular road of diameter 20 m with uniform speed 10 ms^{-1}. A bob is suspended from the roof of the car by a light string of length 1 m. What is the angle made by the string with the road ?

a. 0^0 b. 30^0
c. 45^0 d. 90^0

6. A bob is suspended from an ideal string of length *l*. Now it is pulled to a side till the string makes an angle 60° to the vertical and whirled along a horizontal circle. Then its period of revolution is

a. $\pi\sqrt{\frac{l}{g}}$ b. $\pi\sqrt{\frac{l}{2g}}$

c. $\pi\sqrt{\frac{2l}{g}}$ d. $2\pi\sqrt{\frac{l}{g}}$

7. Consider a conical pendulum with a bob of mass m = 80.0kg on a string of length L= 10.0m that makes an angle of $\theta = 5$.

a. 667 N, 0.857 m/s^2
b. 884 N, 0.657 m/s^2
c. 499 N, 0.612 m/s^2
d. 787 N, 0.857 m/s^2

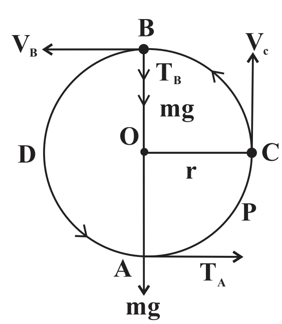
8. A cylindrical drum 1.5 m in diameter and 3 m in height is full of water. The water is emptied into another cylindrical tank in which water rises by 2 m. Find the diameter of the second cylinder up to 2 decimal places.

a. 1.74 m b. 1.94 m
c. 1.64 m d. 1.84 m

1.4 Vertical Circular motion

★Q.12 Using the principle of energy conservation, derive the expression for the minimum speed at different location along a virtical circular motion controlled by gravity.

Ans:



i. When body or particle perform circular motion in vortical plane it posses both types of energies
a. Kinetic energy
b. Potential energy
Total energy at any point is sum of kinetic energy and potential energy.

ii. Consider a body of mass *m*, tied at the end of string. It is whirled in a vortical circle of radius *r*. Let, A, B, and C be points on the vertical and horizontal diameter of circle.
Let, T_A - tension at the lowest point
T_B - tension at the highest point
T_C - tension at horizontal position
Tensions in the string always act along the radius of circular path towards the centre O. The velocities at any point on the circle are tangential to the circular path.

iii. When particle moves from bottom to top, Kinetic energy is converted into potential energy. and vice versa while coming back.

Expression for velocities :

i. Velocity at B : At the highest point B, the tension in the string together with weight provides necessary centripetal force.

$$\therefore \quad T_B + mg = \frac{m v_B^2}{R}$$

At the point B, $T_B = 0$.
Therefore the weight only provides necessary centripetal force.

$$\therefore \quad mg = \frac{m v_B^2}{R}$$

$$\therefore \quad v_B = \sqrt{gr} \quad \text{... (1)}$$

The string will not slack if the minimum velocity at the top is greater than or equal to $\sqrt{gr}$.

ii. Velocity at A :
The energy possessed by a body at point A is totally kinetic energy.

$$\begin{pmatrix}\text{K.E.}\\ \text{at A}\end{pmatrix} = \begin{pmatrix}\text{K.E.}\\ \text{at B}\end{pmatrix} + \begin{pmatrix}\text{Change in P.E.}\\ \text{incoming from B to A}\end{pmatrix}$$

$$\therefore \quad \frac{1}{2}mv_A^2 = \frac{1}{2}mv_B^2 + mgh$$

$v_A^2 = v_B^2 + 4gr$ but $h = 2r$

$v_A^2 = gr + 4gr = 5gr$

$$\therefore \quad v_A = \sqrt{5gr} \quad \text{...(2)}$$

This equation given the magnitude of velocity at the lowest point.

iii. Velocity at point C :

$$\begin{pmatrix}\text{K.E.}\\ \text{at A}\end{pmatrix} = \begin{pmatrix}\text{K.E.}\\ \text{at C}\end{pmatrix} + \begin{pmatrix}\text{Change in P.E.}\\ \text{incoming from A to C}\end{pmatrix}$$

$\therefore \quad \frac{1}{2} mv_A^2 = \frac{1}{2} mv_C^2 + mgh$

$v_A^2 = v_C^2 + 2gr$

but h = r

$v_C^2 = v_A^2 - 2gr$

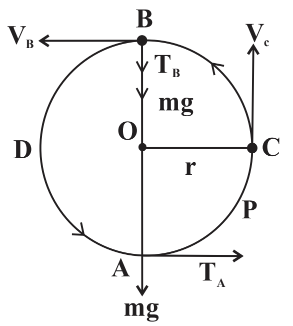
$= 5gr - 2gr$

$\therefore \quad v_C = \sqrt{3gr}$ .... (3)

The velocity at point C and D are equal.

$\therefore \quad v_D = \sqrt{3gr}$

Note:



Velocity at point P :

$$\begin{pmatrix}\text{K.E.}\\ \text{at A}\end{pmatrix} = \begin{pmatrix}\text{K.E.}\\ \text{at P}\end{pmatrix} + \begin{pmatrix}\text{Change in P.E.}\\ \text{incoming from A to P}\end{pmatrix}$$

$\therefore \quad \frac{1}{2} mv_A^2 = \frac{1}{2} mv_P^2 + mgh_P$

Multiply by $\frac{2}{m}$,

$v_A^2 = v_P^2 + 2gh_P$

but from figure,

h = r(1–cosq)

$\therefore \quad v_A^2 = v_P^2 + 2gr\ (1 - \cos\theta)$

$v_P^2 = v_A^2 - 2gr\ (1 - \cos\theta)$

$v_P = \sqrt{v_A^2 - 2gr(1 - \cos\theta)}$

As, $\quad v_A = \sqrt{5gr}$

$v_P = \sqrt{5gr - 2gr(1 - \cos\theta)}$

$v_P = \sqrt{3gr + 2gr\cos\theta}$

$\therefore \quad \boxed{v_p = \sqrt{gr\ (3 + 2\cos\theta)}}$

Q.13 Show that, difference in tension of lowest and highest point in v.c.m. is 6 mg.

Or

In the vertical circular motion of a body contolled by gravity, prove that the diffrence between the extreme tension (or normal force) depends only upon the weight of the body.

Ans:

i. In v.c.m., tension at lowest point (T_A).

$T_A = \frac{mv_A^2}{r} + mg$ ...(1)

ii. Tension at highest point (T_B),

$T_B = \frac{mv_B^2}{r} - mg$ ...(2)

iii. Substracting eqution (2) from equation (1),

$T_A - T_B = \frac{mv_A^2}{r} + mg - \left[\frac{mv_B^2}{r} - mg\right]$

$= \frac{m}{r}\left[v_A^2 - v_B^2\right] + 2mg$ ...(3)

iv. From III[rd] kinematical equation of linear motion

$v^2 = u^2 + 2as$

Substituting for lowest and highest point.

$u = v_A$, $v = v_B$,

$a = -g$, $\quad s = 2r$

$\therefore \quad v_B^2 = v_A^2 + 2(-g)\ 2r$

$4\ gr = v_A^2 - v_B^2$

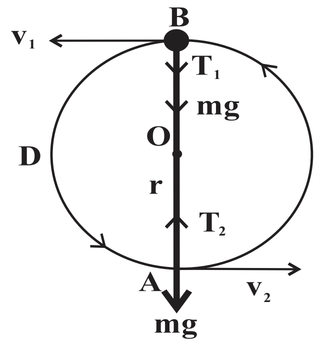
v. Substituting in Eq. (3)

$T_A - T_B = \frac{m}{r}(4gr) + 2mg = 6mg$

Q.14 A body at the end of rod is revolved in non uniform vertical circular motion then show that the difference in tension in the rod at the heighest and lowest position is 6mg.

Ans.

i. Consider a body of mass m attached to a rod and revolved in a vertical circle of radius r at a place where the acceration due to gravity is g. <u>We shall assume that the rod is not rigid so that the tension in the rod changes.</u> Let T_1 be the tension at Top position and T_2 tension at lowest position.



i Let v_1 and v_2 be the speeds at the highest point A and lowest point B.

ii. At the top, both T_1 and weight mg are vertically downward.

$\therefore$ $T_1 + mg = \frac{mv_1^2}{r}$

iii. Taking the minimum value of $v_1 = 0$,

$T_1 + mg = 0$ ...(1)

iv. The total energy at the top

$= KE + PE$

$= \frac{1}{2}mv_1^2 + mg(2r)$

$= 2mgr \quad (\therefore v_1 = 0)$ ...(2)

v. At the bottom,

$T_2 - mg = \frac{mv_2^2}{r}$ ...(3)

vi. and the total energy = KE + PE

$= \frac{1}{2}mv_2^2 + 0 = \frac{1}{2}mv_2^2$ ...(4)

vii. According to law of conservation of energy Total energy at bottom = Total energy at Top

$\frac{1}{2}mv_2^2 = 2mgr$

$\therefore$ $v_2^2 = 4gr$

$\therefore$ $v_2 = 2\sqrt{gr}$ ...(5)

viii. Substituting for v_2 in equation(3)

$T_2 - mg = \frac{m}{r}(4gr) = 4mg$

$\therefore$ $T_2 = 5mg$ ...(6)

ix. Subtracting equation (1) and (6)

$T_2 - (T_1 + mg) = 5mg$

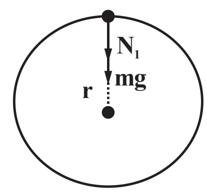
$\therefore$ $T_2 - T_1 = 6$ mg

Hence proved

Q.15 Give reason : Motorcyclist driving in a vertical loops inside a sphere of death does not fall down when at the heighest point of the chamber.

Ans. A motorcyclist driving in vertical loops inside a hollow glove performs vertical circular motion.

Suppose the mass of the motorcycle and motorcyclist is m and the radius of the chamber is r.



ii. At the highest point Normal (N_1) is acting downward and weight (mg) is also acting downward

$\therefore$ Net force acting on motercyclist = N_1 + mg

iii. If this forceis able toprovide the necessary centripetal force at the highest point, the motorcycle does not lose contact with the glove and fall down. The minimum value of this force is found from the limiting case when N_1 just becomes zero and the weight alone provides the necessary centripetal force:

$\frac{mv_1^2}{r} = mg$

$\therefore$ $v_1^2 = gr$ or $v_1 = \sqrt{gr}$

This is the required minimum speed at highest point. So that motorcyclist should not lose contact at top most point.

Q.16 Obtain an expression for upper speed limit of a vehicle at the top of a convex overbridge

Ans.

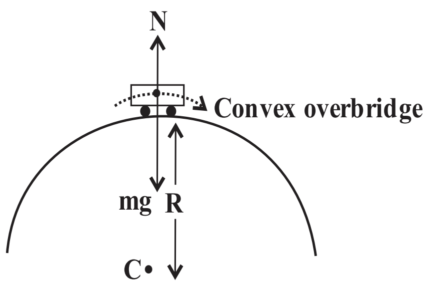
i. Suppose a car of mass m, travelling with a uniform speed v, acrosses over a bridge which is in the form of a convex arc of radius r.

ii. At heighest point, Normal is acting upward and weight is acting downward

$\therefore$ Net force = mg – N

This force provides necessary Centripetal force

$$\therefore \quad mg - N = \frac{mv^2}{r} \quad ...(1)$$



$$\therefore \quad mg - N = \frac{mv^2}{r}$$

$$\therefore \quad N = m\left(g - \frac{v^2}{r}\right) \quad ...(1)$$

is the required expression. It shows that as v increases, N decreases.

iii. For just maintainins contact, N = 0

There equation (1) becomes,

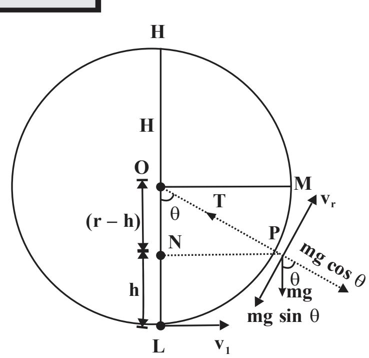
$$\cancel{m}g = \frac{\cancel{m}v^2}{r}$$

$$v^2 = rg$$

$$\boxed{v = \sqrt{rg}}$$

Therefore, this is the maximum speed with which a car can cross the bridge, irrespective of its mass.

Key Points



i. The velocity of the stone is minimum at the highest point and maximum at the lowest point.

ii. **Velocity in vertical circular motion:**

a. Velocity of the particle at the lowest position is maximum and it is,

$$v_L = \sqrt{5gr}$$

b. Velocity of the particle at the top position is minimum and it is,

$$v_T = \sqrt{gr}$$

c. Velocity of the particle at the horizontal position is,

$$v_M = \sqrt{3gr}$$

d. Velocity of the particle at any position is,

$$v_P = \sqrt{v_l^2 - 2gh} = \sqrt{v_l^2 - 2gl(1-\cos\theta)}$$

$$= \sqrt{gr(3+2\cos\theta)}$$

Velocity varies position to position and it is non conservative.

iii. The instantaneous tension in the string.

$$T = \frac{mv_P^2}{r} + mg\cos\theta$$

$$T = 3\,mg\,[1 + \cos\theta]$$

iv. Minimum tension at lowest, horizontal and heighest position is

$T_L = 6$ mg, $T_M = 3$ mg and $T_H = 0$

v. Kinetic energy at any position of the particle in vertical circular motion is,

$$KE. = \frac{1}{2}\,m\,v_p^2 = \frac{1}{2}\,m(3 + 2\cos\theta)\,gr$$

Kinetic energy varies position to position of the particle performing vertical circular motion and it is non conservative.

vi. Potential energy at any position of the particle in vertical circular motion is,

P.E. = mgh = mgr $(1 - \cos\theta)$

Potential energy varies position to position of the particle performing vertical circular motion and it is non conservative.

vii. Total energy of the particle performing vertical circular motion is conservative i.e. remains constant.

$$E = \frac{5}{2} m g r$$

Tips for Solving top Numerical

i. At topmost position, of convex bridge

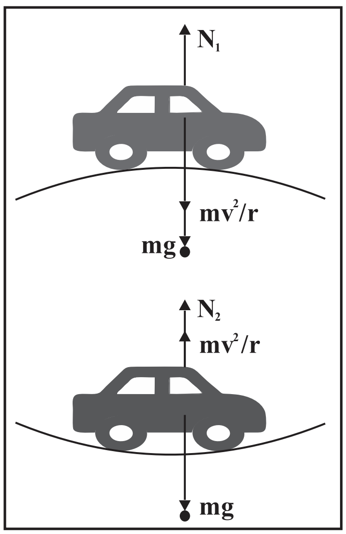
$N_1 < mg$

$\therefore \quad mg - N_1 = \frac{mv^2}{r}$

At Bottom position of concave bridge

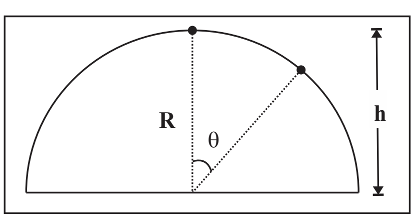
$N_2 > mg$

$N_2 - mg = \frac{mv^2}{r}$

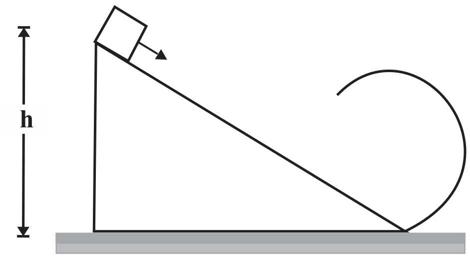


ii. If a bucket filled with water is whirled in a vertical circle at the end of a rope, water will not fall down when it is at the highest point if its velocity at that point is $v \geq \sqrt{gr}$.

iii. A particle begins to slide without any friction from the top of a hemisphere of radius R as shown. It leaves the surface of hemisphere at height 'h' above the centre, such that $h = 2R/3$ and $\cos\theta = 2/3$



iv. When a body slides along an inclined plane of height 'h' and describes a vertical circle of radius 'r' on reaching the bottom, then



$h = 5r / 2.$

$m g h = \frac{1}{2} mv^2$ where $v = \sqrt{5gr}$

$\Rightarrow h = \frac{5r}{2}$

Type - X

Numerical based on vertical circular motion

Formulae used

1. At lowest position, $v_L = \sqrt{5gr}$
2. At midway position $v_M = \sqrt{3gr}$
3. At highest position $v_H = \sqrt{gr}$

1) A stone is whirled in a vertical circle at the end of a rope of length 0.5m. Find the velocity of stone at

i. lowest position

ii. midway position

iii. Top position to just complete the circle. ($g = 9.8\ m/s^2$)

Data: $r = 0.5$ m, $g = 9.8\ m/s^2$

To Find: v_L, v_M, v_H

Formula: At lowest position,

$v_L = \sqrt{5gr}$

At midway position

$v_M = \sqrt{3gr}$

At highest position

$v_H = \sqrt{gr}$

Solution:

At lowest position,

$$v_L = \sqrt{5gr} = \sqrt{5 \times 9.8 \times 0.5} = 4.949 m/s$$

At midway position

$$v_M = \sqrt{3gr} = \sqrt{3 \times 9.8 \times 0.5} = 3.834 m/s$$

At highest position

$$v_H = \sqrt{gr} = \sqrt{9.8 \times 0.5} = 2.213 m/s$$

Ans :	Velocity at lowest position is 4.949 m/s, velocity at midway is 3.834 m/s and velocity at highest position is 2.213 m/s

2) In a circus, motor cyclist having mass of 50 kg move in a spherical cage (Death well) of radius 3m. Calculate the least velocity with which he must pass highest point without loosing. Also calculate his angular speed at highest point.

Data: m = 50 kg, r = 3 m

To Find: v_{min} , ω

Formula: i. Least velocity at top

$$v = \sqrt{rg}$$

ii. v = rw

$$w = \frac{v}{r}$$

Solution:

i. $v = \sqrt{rg} = \sqrt{3 \times 9.8} = 5.421$ m/s

ii. $w = \frac{v}{r} = \frac{5.421}{3} = 1.807$ rad/s

Ans :	The least velocity with which he must pass highest point without loosing is 5.421 m/s and angular velocity is 1.807 rad/s

3) A stone weighing 1 kg is whirled in a vertical circle at the end of a rope of length 0.5 m. Find the tension at
i. Lowest position
ii. Mid position
iii. Highest position

Data: m = 1 kg, r =1 0.5 m, g = 9.8 m/s^2

To find: T_L , T_M , T_H

Formula : i. $T_L = 6$ mg

ii. $T_M = 3$ mg

iii. $T_H = 0$

Solution:

i. $T_L = 6 mg = 6 \times 1 \times 9.8 = 58.8$ N

ii. $T_M = 3 mg = 3 \times 1 \times 9.8 = 29.4$ N

iii. $T_H = 0$

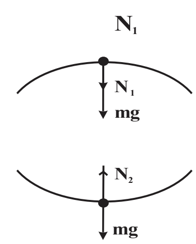
Ans :	Tension at lowest position is 58.8 N, tension at midway is 29.4 N and tension at heighest postion is 0

4) A pilot of mass 50 kg in a jet aircraft executes a "loop - the - loop" with constant speed of 250 m/s. If the radius of the vertical circle is 5 km, compute the force exerted by the seat on the pilot at
i. the top of the loop
ii. the bottom of the loop.

Data: m = 50 kg, v = 250 m/s,
r = 5 km = 5×10^3 m

To find: Force exerted by the seat on the pilot
i. at top of the loop (N_1)
ii at bottom of loop (N_2)

Formula:



i. At the top of the loop :

$$\therefore \quad N_1 + mg = \frac{mv^2}{r}$$

$$\therefore \quad N_1 = m\left(\frac{v^2}{r} - g\right)$$

ii. At the bottom of the loop :

$$\therefore \quad N_2 - mg = \frac{mv^2}{r}$$

$$\therefore \quad N_2 = m\left(\frac{v^2}{r} + g\right)$$

Solution:

i. At the top of the loop :

$$\therefore \quad N_1 = m\left(\frac{v^2}{r} - g\right)$$

$$= 50\left[\frac{(250)^2}{5 \times 10^3} - 9.8\right] = 50\,(12.5 - 9.8)$$

$= 50\,(2.7) = 135\text{ N}$

ii. At the bottom of the loop :

$\therefore \quad N_2 = m\left(\frac{v^2}{r} + g\right)$

$= 50\left[\frac{(250)^2}{5\times10^3} + 9.8\right] = 50\,(12.5 + 9.8)$

$= 50\,(22.3) = 1115\text{ N}$

Ans : The forces exerted by the seat on the pilot at the top and bottom of the loop are 135 N and 1115 N, respectively.

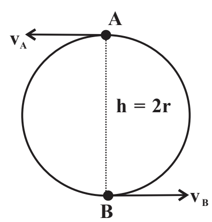
★5) A motorcyclist (as a particle) is undergoing vertica circles inside a sphere of death. The Speed of the motorcycle varies between 6 m/s and 10 m/s. Calculate diameter of the sphere of death. How much minimum values are possible for these two speeds?

Data: $v_B = 10\text{m/s},\ v_A = 6\text{m/s}$

To find: i. Diameter of sphere of death

ii. $(v_B)_{min}$ and $(v_A)_{min}$

Formulae:



i. (K.E + P.E) at point B
= (K.E + P.E) at Point A

$\frac{1}{2}mv_B^2 + 0 = \frac{1}{2}mv_A^2 + mgh$

$v_B^2 = v_A^2 + 2g\,(2r)$

$v_B^2 - v_A^2 = 4rg$

ii. $(v_B)_{min} = \sqrt{5rg}$

iii. $(v_A)_{min} = \sqrt{rg}$

Solution:

i. $v_B^2 - v_A^2 = 4rg$

$(10)^2 - (6)^2 = 4(r)(10)$

$\therefore \quad r = \frac{64}{40} = 1.6\text{m}$

$\therefore \quad d = 2r = 2 \times 1.6\text{m} = 3.2\text{ m}$

ii. $(v_B)_{min} = \sqrt{5rg}$

$(v_B)_{min} = \sqrt{5\times1.6\times10} = 4\sqrt{5}\text{ m/s}$

iii. $(v_A)_{min} = \sqrt{rg}$

$(v_A)_{min} = \sqrt{1.6\times10} = 4\text{ m/s}$

Ans : i. Diameter of the sphere of death is 3.2m.
ii. The min imum velocity at the lowest po int can be $4\sqrt{5}$ m / s while at highest po int, it will be 4 m / s

6) A bucket is tied at the of string 0.9 m long and whirled in a vertical circle with constant speed. What should be the minimum speed so that the water from the bucket does not spill, when the bucket is at the highest position (Take g = 10m/s²)

Data: $r = 0.9\text{ m},\ g = 10\text{ m/s}^2$

To find: Velocity at highest position (v)

Formulae : $v = \sqrt{rg}$

Solution:

$v = \sqrt{0.9\times10} = \sqrt{9}$

$v = 3\text{ m/s}$

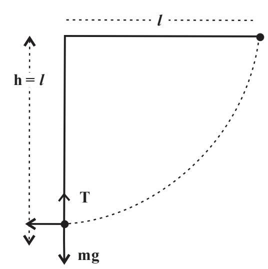
Ans : The velocity of bucket at heighest point is 3m/s

7) A pendulum of mass m and string '*l*' is held in horizontal position and then released into a vertical circle . Find the velocity of bob and tension in string at lowest position

To find: Velocity at lowest position (v)
Tension in string (T)

Solution:

i. To find velocity at lowest position



According to law of conservation of

energy

$\frac{1}{2}\cancel{m}v^2 = \cancel{m}gh$

$v^2 = 2gl$

$\therefore \quad \boxed{v = \sqrt{2gl}}$

ii. From diagram

$T - mg = \frac{mv^2}{r}$

$T - mg = \frac{m\left(\sqrt{2g\,l}\right)^2}{l}$

$T = mg + \frac{m \times 2g\,\cancel{l}}{\cancel{l}}$

$\boxed{T = 3mg}$

Ans : At lowest position velocity is $\sqrt{2gl}$ and tension is 3mg

Problem for Practice

1. An object of mass 1 kg is tied to one end of a string of length 9 m and whirled in a vertical circle. What is the minimum speed required at the lowest positon to complete a circle?

Ans : 8.445 m/s

2. A stone of mass one kilogram is tied to the end of a string of length 5 m and whirled in a vertical circle. What will be the minimum speed required at the lowest position to complete the circle? (Given: $g = 9.8\ m/s^2$)

Ans : $7 ms^{-1}$

3. A stone of mass kg. ties to one end of rope of length 0.8m, is whirled in a vertical circle. Find the minimum velocityat the highest point and at the midway point. [$g = 9.8\ m/s^2$]

Ans : 2.8 m/s, 4.85 m/s

4. A stoneof mass 100 g attached to a string of length 50 cm is whirled in a vertical circle by giving velocity at lowest point as 7 m/s. Find the velocity at the highest point. [Acceleration due to gravity = $9.8\ m/s^2$]

Ans : 5.422 m/s

5. A bucket containing some water is tied to one end of a rope 0.8 m long and rotated about the other end in a vertical circle. Find the minimum number of rotations the bucket can make per minute in order that water in the bucket may not spill. ($g = 9.8\ m/s^2$) What is the maximum speed at the topmost position?

Ans: 33.44 r.p.m.; 2.8m/s

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 6)

1. A bucket tied at the end of a 1.6 m long string is whirled in a vertical circle with constant speed. What should be the minimum speed so that the water from the bucket does not spill, when the bucket is at the highest position (Take

a. 4 m/sec b. 6.25 m/sec
c. 16 m/sec d. None of the above

2. A 1 kg stone at the end of 1 m long string is whirled in a vertical circle at constant speed of 4 m/sec. The tension in the string is 6 N, when the stone is at ($g = 10\ m/sec^2$)

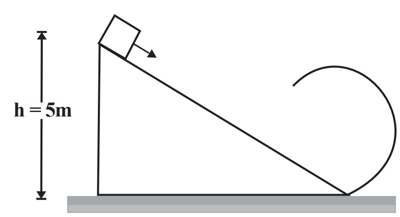
a. Top of the circle
b. Bottom of the circle
c. Half way down
d. None of the above

3. A pendulum bob on a 2 m string is displaced 60^o from the vertical and then released. What is the speed of the bob as it passes through the lowest point in its path

a. $\sqrt{2}\ m/s$ b. $\sqrt{9.8}\ m/s$
c. 4.43 m/s d. $1/\sqrt{2}\ m/s$

4. As per given figure to complete the circular loop what should be the radius if initial height is 5 m

a. 4 m b. 3 m
c. 2.5 m d. 2 m



5. A simple pendulum has a length l. What minimum velocity should be imparted to its bob at the mean position so that the bob reaches a height equal to *l* above the point of suspension?

a. $\sqrt{gr}$ b. $\sqrt{5gl}$

c. $\sqrt{2gl}$ d. $\sqrt{\frac{l}{g}}$

6. A motor cyclist rides in a hollow sphere in a vertical circle of radius 30 m. What will be the minimum speed required so that he does not lose contact with the surface of sphere at the highest point? ($g = 10\ m/s^2$)

a. 5.442 km/s b. 17.422 cm/s

c. 17.32 m/s d. 54.22 m/s

7. A body rests on the top of a hemisphere of radius R. What will be the least horizontal velocity imparted to it, if it has to leave the hemisphere without sliding down?

a. $\sqrt{2gR}$ b. $\sqrt{5gR}$

c. $\sqrt{gR}$ d. $\sqrt{3gR}$

8. The vertical section of a road over the bridge is in the form of circle of radius 15.5 m. What will be the maximum velocity with which a car, whose centre of gravity is 0.5 m above the ground can cross the bridge without losing contact with the surface of bridge at the highest point? ($g = 9.8\ m/s^2$)

a. 10.56 m/s b. 12.56 m/s

c. 14.56 m/s d. 1.256 m/s

9. If a body of mass 0.1 kg tied with a string of length 1 m, is rotated in vertical circle, then the energy of the body at the highest position will be ($g = 9.8\ m/s^2$)

a. 2.45 J b. 1.25 J

c. 3.45 J d. 4.45 J

Try yourself

10. A cane filled with water is revolved in a vertical circle of radius 4 meter and the water just does not fall down. The time period of revolution will be

a. 1 sec b. 10 sec c. 8 sec d. 4 sec

11. A 2 kg stone at the end of a string 1 m long is whirled in a vertical circle at a constant speed. The speed of the stone is 4 m/sec. The tension in the string will be 52 N, when the stone is

a. At the top of the circle

b. At the bottom of the circle

c. Halfway down

d. None of the above

12. If the section of a bridge on a river is an arc of a circle of radius 88.2 m, then the maximum speed with which a car can travel over the bridge without losing contact with the ground level will be

a. 29.4 m/s b. 9 m/s

c. 81 m/s d. 18 m/s

13. A pendulum bob on a 2 m string is displaced 60° from the vertical and then released. What is the speed of the bob as it passes through the lowest point in its path?

a. $\sqrt{2}$ m/s b. $\sqrt{9.8}$ m/s

c. 4.43 m/s d. $1/\sqrt{2}$ m/s

14. Length of a simple pendulum is 2 m and mass of its bob is 0.2 kg. If the tension in the string exceeds 4 N, it will break. If $g = 10\ m/s^2$ and the bob is whirled in a horizontal plane, the maximum angle through which the sting can make with vertical during rotation is

a. 30° b. 45° c. 60° d. 90°

15. In a cylindrical well of death, a motor cyclist rides around the inner wall in horizontal circles. If the diameter of well of death is 18 m, then the minimum speed of cyclist, so as to prevent him from sliding down will be
(u = 0.8 and $g = 10\ m/s^2$)

a. 1.06 m/s b. 10.6 km/s

c. 0.106 m/s d. 10.6 m/s

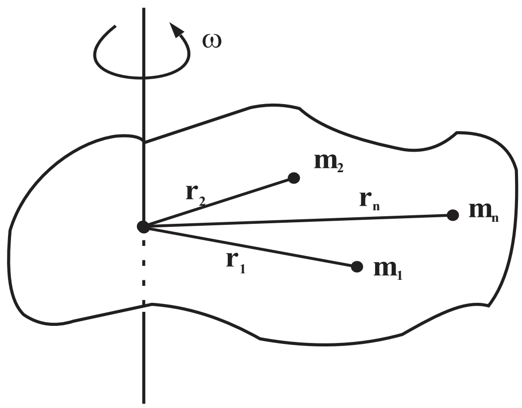
16. In a well of death, motor cycle rider drives round the inner wall of a hollow cylindrical chamber. If the radius of the 'cylindrical chamber is 8 m. What would be minimum speed of the rider to prevent him from sliding down? ($g = 10\ m/s^2$, $\mu = 0.2$)

a. 10 m/s b. 20 m/s c. 30 m/s d. 40 m/s

1.5 Moment of inertia as an Analogous quantity for mass

Q.17 Explain moment of inertia. Give it's equation and units.

Ans: **Moment of inertia :** The Moment of inertia of a body about a given axis of rotation is defined as the sum of the products of the mass of each particle of the body and the square of it's distance from the axis of rotation.



i. Consider a rigid body rotating about a fixed axis passing through point O.

ii. Suppose that the body consists of n particles of masses m_1, m_2, m_3, , ... m_n situated at perpendicular distances r_1, r_2, r_3, ... r_n respectively from the given axis, of rotation.

M.I.= $m_1\ r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ---- + m_n r_n^2$

$$\boxed{M.I. = \sum_{i=1}^{n} m_i r_i^2}$$

iv. If we are to find the moment of inertia of a continuous mass distribution, then we consider an element of mass dm at a distance r from the axis of rotation. The moment of inertia of this infinitesimal element is dI, given by

$dI = r^2 dm$

$$\boxed{I = \int dI = \int r^2 dm}$$

iii. **SI unit :** kg m^2
CGS unit : g cm^2
Dimension : $[M^1 L^2 T^0]$.

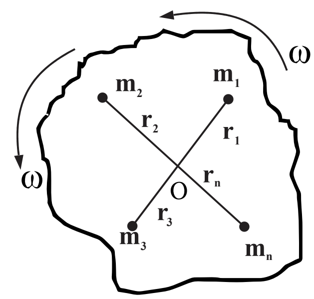
Key Point

i. The value of 'I' depends upon
 a. mass of particles,
 b. their distance from the axis of rotation
 c. position of the axis of rotation.
ii. The moment of inertia of the particles, through which the axis of rotation passes is zero.

Q.18. Obtain an expression for the kinetic energy of a body rotating with a uniform angular velocity about a fixed axis.

Ans:

i. Consider a rigid body rotating with a uniform angular velocity ω about an fixed axis passing through point O.



ii. Suppose that the body consists of n particles of masses $m_1, m_2, m_3, \ldots, m_n$ situated at perpendicular distances r_1, r_2, r_3 ....., r_n respectively from the given axis of rotation . As the body rotates, all the particles perform uniform circular motion with the same angular velocity ω .

iii. The particle of mass m_1 is performing uniform circular motion with angular velocity $\vec{\omega}$ and radius r_1.

∴ linear velocity, $v_1 = r_1\ \omega$

∴ kinetic energy of the first particle,

$$E_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 r_1^2 \omega^2$$

iv. Similarly, K.E. of the second particle,

$$E_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 r_2^2 \omega^2$$

similarly, K.E. of the third particle,

$$E_3 = \frac{1}{2} m_3 v_3^2 = \frac{1}{2} m_3 r_3^2 \omega^2$$

K.E. of the n^{th} particle,

$$E_n = \frac{1}{2}m_n v_n^2 = \frac{1}{2}m_n r_n^2 \omega^2$$

v. Kinetic energy E of rotating body, is given by

$$E_{rot} = E_1 + E_2 + E_3 + ... + E_n$$

$$= \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \frac{1}{2}m_3 r_3^2\omega^2 + ... + \frac{1}{2}m_n r_n^2\omega^2$$

$$= \frac{1}{2}[\, m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... + m_n r_n^2 \,]\,\omega^2$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} m_i r_i^2\right]\omega^2$$

But, $\sum_{i=1}^{n} m_i r_i^2 = I$ is the moment of inertia of the body about a given axis of rotation.

$$\text{K.E. of rotating body} = \frac{1}{2}I\omega^2$$

Q.19 Give physical significance of Moment of inertia:

Ans:

i. In translational motion,

$$\text{K.E. of body} = \frac{1}{2}mv^2 \qquad ...(1)$$

In rotational motion,

$$\text{K.E. of rotating body} = \frac{1}{2}I\omega^2 \qquad ...(2)$$

ii. By comparison of these two expressions, it is clear that moment of inertia in rotational motion must be analogous to mass in translational motion.

iii. In translational motion, the inertia of a body represents its reluctance to undergo a change in its state of translational motion. Similarly, moment of inertia in rotational motion represents the reluctance of the body to undergo a change in its state of rotation. Thus, moment of inertia is inertia in rotational motion.

Type - XI
Numerical based on K.E of rotating body

Formulae

1. K.E. of rotating body $= \frac{1}{2}I\omega^2$

2. work done = change in kinetic energy.

work done = $\Delta K.E_{final} - \Delta K.E_{initial}$

1) An energy of 500 J is spent to increase the speed of wheel from 60 rpm to 240 rpm. Calculate the moment of inertia of the wheel.

Data: $\omega_1 = 60 \text{ rpm} = \frac{60}{60} \times 2\pi = 2\pi$ rad/s

$\omega_2 = 240 \text{ rpm} = \frac{240}{60} \times 2\pi = 8\pi$ rad/s

To find: Moment of inertia (I) W = 500 J

Formula:

work done = change in kinetic energy.

work done = $\Delta K.E_{final} - \Delta K.E_{initial}$

$$W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

$$W = \frac{1}{2}I\left(\omega_2^2 - \omega_1^2\right)$$

Solution:

$$W = \frac{1}{2}I\left(\omega_2^2 - \omega_1^2\right)$$

$$\therefore \quad 500 = \frac{1}{2}I\,(64\pi^2 - 4\pi^2)$$

$$500 = \frac{1}{2}I\,(60\pi^2)$$

$$\therefore \quad I = \frac{1000}{60\pi^2} = \frac{100}{6\pi^2} = 1.688 \text{ kg m}^2$$

Ans : The Moment of inertia of body is 1.688 kgm^2

2) The frequency of rotation of a disc of M.I. 0.04 kg m² changes from 60 rpm to 180 rpm in 20 sec. Find its angular acceleration and the workdone in this time.

Data: $\omega_0 = 60 \text{ rpm} = 60 \times \frac{2\pi}{60} = 2\pi$ rad/sec

$\omega = 180 \text{ rpm} = 180 \times \frac{2\pi}{60} = 6\pi$ rad/sec

To find: i. Angular acceleration (α)

I = 0.04 kgm^2 t = 20 sec

ii. Work done

Formula :

i. $\omega = \omega_0 + \alpha t$

$\therefore \quad \alpha = \frac{\omega - \omega_0}{t}$

ii. work done = change in kinetic energy.

work done = $\Delta K.E_{final} - \Delta K.E_{initial}$

$$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

$$W = \frac{1}{2}I\left(\omega^2 - \omega_0^2\right)$$

Solution:

i. $\omega = \omega_0 + \alpha t$

$\therefore$ $\alpha = \frac{\omega - \omega_0}{t} = \frac{6\pi - 2\pi}{20} = \frac{4\pi}{20} = \frac{\pi}{5}$ rad/sec^2

ii. $W = \frac{1}{2}I\left(\omega^2 - \omega_0^2\right)$

$= \frac{1}{2} \times 0.04 \times \left(6\pi^2 - 2\pi^2\right)$

$= 0.02 \times 4\pi^2 = 0.8$ J

Ans:	The angular acceleration is $\pi/5$ rad/s^2 and work done in this time is 0.8 J

Problem for Practice

1. A rotating wheel of moment of inertia 0.02 kgm^2 changes angular speed from 1800 rpm to 3000 rpm in 20 s. What is the angular acceleration and work done in this time.

Ans: 2π rad/s,4 J

2. A wheel is rotating at a rate of 1000 rpm and its kinetic energy is 10^6 J. Determine the moment of inertia of the wheel about its axis of rotation.

Ans: 182.4 kg m^2

3. Energy of 484 J is spent in increasing the speed of a flywheel from 60 rpm to 360 rpm Find the moment of inertia of the wheel.

Ans: 0.7 kg m^2

1.6 Radius of Gyration

Q.20 Define Radius of gyration. Give its physical significance.

Ans: Radius of gyration :

i. **Defination :** The radius of gyration of a body about a given axis of rotation is defined as the distance between the axis of rotation and the point at which the whole mass of the body can be supposed to be concentrated so as to possess the same moment of inertia as that of the body.

$I = MK^2$

$\therefore$ $K = \sqrt{\frac{I}{M}}$

Physical significance of radius of gyration:

i. The radius of gyration of a rigid body about a given axis depends upon distribution of the mass about the axis of rotation.

ii. We have $I = MK^2$

The radius of gyration has the least value when the axis of rotation passes through the centre of mass of the body.

iii. The radius of gyration increases as the distance of the axis of rotation from the centre of mass increases.

Thus, the gyration is a measure of the distribution of the mass of the rotating body around the axis of rotation.

★ Q.21 Discuss the necessity of radius of gyration.On what factor does it depend?

Or

★ Why is it useful to define radius of gyration?

Ans:

i. Experimentally we can determine the moment of inertia of any object. It depends upon mass of that object and how that mass is distributed from or around the given axis of rotation.

ii. If we are interested in knowing only the mass distribution around the axis of rotation, we can express moment of inertia of any object as $I = MK^2$

where M is mass of that object.

iii. It means that the mass of that object is effectively at a distance K from the given axis of rotation.

In this case, K is defined as the radius of gyration of the object about the given axis of rotation.

iv. The radius of gyration increases as the distance of the axis of rotation from the centre of mass increases. Thus, the gyration is a measure of the distribution of the mass of the rotating body around the axis of rotation.

v. The radius of gyration of a body about a given axis depends upon distribution of the mass about the axis of rotation.

Key Point

i. The radius of gyration of a body is not a constant quantity. Its value changes with change of position of its axis of rotation.

1.7 Theorem of parallel axes and Theorem of perpendicular axes

Q.22 State and prove principle of parallel axes.

Ans: Principle of parallel axes :

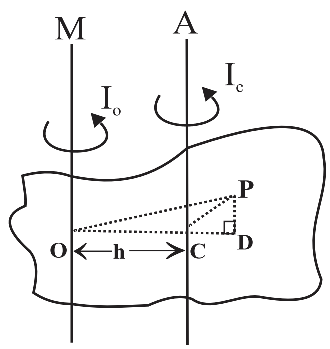
The moment of inertia of a body about an axis is equal to the sum of its moment of inertia about a parallel axis passing through its centre of mass and the product of the mass of the body and the square of the distance between the two parallel axes.

Proof :

i. Consider, a rigid body rotating about a fixed axis passing through point O.

ii. Let C be the centre of mass and OC = h. The axes through O and C are taken parallel.

iii. Imagine a small element of mass dm of the body situated at point P. Join PO, PC and draw perpendicular PD on the line OC (produced).



iv. M.I. about an axis passing through the point O,

a. For element = OP^2 dm.

b. For body $I_o = \int OP^2\, dm$ ... (1)

v. M.I. about the axis passing through point C,

a. For element = CP^2 dm

b. For body $I_c = \int CP^2\, dm$ ... (2)

vi. In ΔOPD, by pathgoras theorem

$OP^2 = (OD^2 + PD^2)$

$= [(OC + CD)^2 + PD^2]$

$= (OC^2 + 2OC.CD + CD^2 + PD^2)$

$OP^2 = (h^2 + 2h.CD + CP^2)$ ... (3)

$(\because CD^2 + PD^2 = PC^2)$

vii. M.I. about the axis passing through point O.

$I_0 = \int OP^2 dm$ [From (1)]

$= \int (CP^2 + h^2 + 2h.CD) dm$ [From (3)]

$I_0 = \int CP^2\, dm + h^2 \int dm + 2h \int CD\, dm$ (4)

viii. But, $\int CP^2\, dm = I_c$

$\int dm = M$ is the mass of body.

CD is the distance of a point from the centre of mass. Any mass distribution is symmetric about the centre of mass

Thus, from the defination of centre of mass,

$\int CD\, dm = 0$

ix. Substituting all values in equation (4)

$\therefore$ $I_o = I_c + h^2 M = I_c + Mh^2$

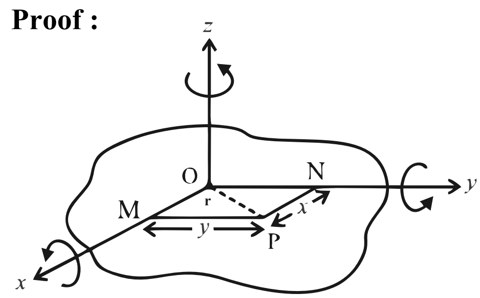
Thus, the principle of parallel axes is proved.

Q.23 State and prove principle of perpendicular axes.

Ans: Principle of perpendicular axes :

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two mutually perpendicular axes in the plane of the lamina and intersecting at the point where the perpendicular axis cuts the lamina.

Proof :



i. Consider a lamina in the horizontal (X-Y) plane. OX and OY are two mutually perpendicular axes in the plane of lamina, while the third axis OZ is perpendicular to the plane of the lamina.

ii. Imagine a small element of the lamina of mass dm situated at the point P (x, y). Draw perpendiculars from P on X and Y and join OP.

iii. M.I. of the element about X-axis is y^2 dm.

∴ M.I. of the lamina about X- axis,

$$I_x = \int y^2 dm \quad \text{... (1)}$$

iv. M. I. of the element about Y - axis is x^2dm

∴ M.I. of the lamina about Y- axis,

$$I_y = \int x^2 dm \quad \text{... (2)}$$

v. M.I. of the element about Z-axis is OP^2 dm.

∴ M.I of the lamina about Z - axis,

$$I_z = \int OP^2 dm \quad \text{... (3)}$$

vi. Now adding equation (1) and (2) ,

$$I_x + I_y = \int y^2 dm + \int x^2 dm$$

$$= \int (y^2 + x^2) dm = \int OP^2 dm = I_z$$

∴ $I_z = I_x + I_y$

Thus, the principle of perpendicular axes is proved.

★ Q.24 State the conditions under which the theorems of parallel axes and prependicular axes are applicable. State the respective mathematical expressions

i. In order to apply parallel axes theorem to any object, we need two axes parallel to each other with one of them passing through the centre of mass of the object.

ii. Perpendicular axes theorem relates the moment of inertia of a laminar object about three mutually perpendicular and concurrent axes two of them in the plane of the object and the third perpendicular to the object.

iii. Mathematical expressions :

a. Parallel axes theorem : $I_o = I_c + Mh^2$

b. Perpendicular axes theorem : $I_z = I_x + I_y$

Type - XII
Numerical based on parallel axis and perpendicular axis

Formulae

1. $I_z = I_x + I_y$
2. $I_o = I_c + h^2 M = I_c + Mh^2$

1) The radius of gyration of a body about an axis at a distance of 6cm from its centre of mass is 10cm. Find the radius of gyration of the body about a parallel axis passing through its centre of mass.

Data: h = 6 cm; K_o = 10 cm

To Find: K_c

Formula i. $I_0 = I_c + Mh^2$

ii. $I = MK^2$

Solution:

According to the principle of parallel axes,

$I_0 = I_c + Mh^2$

∴ $MK_0^2 = MK_c^2 + Mh^2$

∴ $K_0^2 = K_c^2 + h^2$

∴ $K_c^2 = K_0^2 - h^2 = (10)^2 - (6)^2$

$= 100 - 36 = 64$

∴ $K_c = 8$ cm

Ans : Radius of gyration is 8 cm

Problem for Practice

1. Radius of gyration of a body about an axis at a distance of 12 cm from its centre of mass is 0.13 m. Find its radius of gyration about a parallel axis through the centre of mass.

Ans : 5 cm

2. The radius of gyration of a body about an axis at a distance of 4cm from its centre of mass is 5cm. Find the radius of gyration of the body about a parallel axis passing through its centre of mass.

Ans : 3 cm

☞ (A) Rod

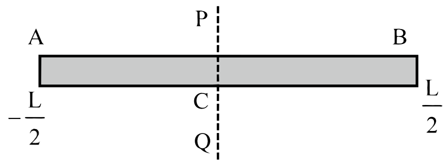
Q.25 State an expression for moment of inertia of a thin uniform rod

A) about an axis passing through its centre of mass and perpendicular to its length.

B) about an axis passing through its one

end and perpendicular to its length.

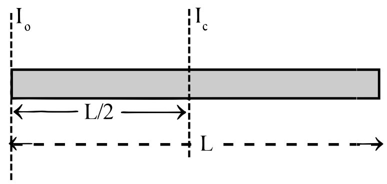
Ans: Consider a thin uniform rod AB of length L and mass M.



A) M.I of Rod about an axis of rotation passing through its centre of mass C and perpendicular to its length.

$$\boxed{I = \frac{ML^2}{12}}$$

B) About an axis passing through its one end and perpendicular to its length.



Using parallel axes theorem,

$I_o = I_c + Mh^2$

$$I_o = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{ML^2}{12} + \frac{ML^2}{4}$$

$$\therefore \quad \boxed{I_0 = \frac{ML^2}{3}}$$

Type - XIII

Numerical based on M.I of Rod

Formulae:

1. M.I of a rod about an axis passing from centre and perpendicular to length

$$I = \frac{ML^2}{12}$$

2. Radius of gyration

$I = MK^2$

$$\therefore \quad K = \sqrt{\frac{I}{M}}$$

3. Parallel axes theorem

$I_0 = I_c + Mh_2$

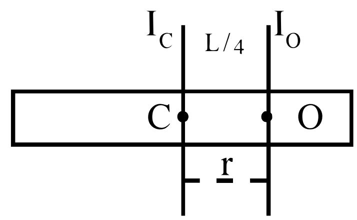
4. Perpendicular axis theorem

$I_z = I_x + I_y$

1) Obtain an expression for moment of inertia of a thin rod about an axis passing through mid point of its one end and center of mass and axis is perpendicular to its length. Also obtain it's coresponding radius of gyration.

Solution:

i. M.I. of the rod about its c.m. and axis perpendicular to its length = $I_c = \frac{ML^2}{12}$



ii. Using parallel axes theorem,

$I_o = I_c + Mh^2$

$$I_o = \frac{ML^2}{12} + M\left(\frac{L}{4}\right)^2$$

$$= \frac{ML^2}{12} + \frac{ML^2}{16}$$

$$\therefore \quad \boxed{I_0 = \frac{7ML^2}{48}}$$

iii. Radius of gyration,

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{7ML^2/48}{M}} = \frac{L}{4}\sqrt{\frac{7}{3}}$$

2) A thin rod has mass 1 kg and length 2 m. Find its M.I. about a transverse axis through a point at a distance of 1 m from the centre of the rod.

Data: $h = 1\text{m}, \ M = 1\text{kg}, \ L = 2\text{ m}$

To Find : Moment of inertia of rod about axis passing 1m from centre of mass (I)

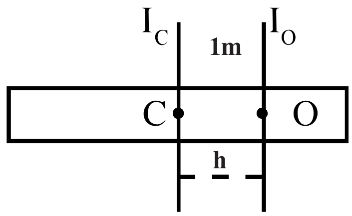
Formula: i. $I_c = \frac{ML^2}{12}$

ii. $I_0 = I_c + Mh^2$

Solution:

i. M.I. of thin rod about on axis passing through centre and perpendicular it is,

$$I_c = \frac{ML^2}{12}$$



ii. By principle of parallel axes,

$I_O = I_c + Mh^2$

$$I_O = \frac{ML^2}{12} + Mh^2 = M\left[\frac{L^2}{12} + h^2\right]$$

$$I_O = 1 \times \left[\frac{(2)^2}{12} + 1\right] = \frac{1}{3} + 1 = \frac{4}{3}$$

$\therefore$ $I_O = 1.33 \text{ kg m}^2$

Ans :	Moment of inertia of rod about axis passing from 1m from centre is 1.33 kgm^2

Problem for Practice

1. A thin uniform rod of length 1 m and mass 1 kg is rotating about an axis passing through its centre and perpendicular to its length. Calculate moment of inertia and radius of gyration of the rod about an axis passing through a point mid way between the centre and its edge, perpendicular to its length.

Ans: 0.1458 kg.m², 0.3818 m

2. Calculate the M.I. of a thin uniform rod of mass one kg and length 60 cm about an a×is perpendicular to its length and passing through (1) its centre (2) its one end

Ans: 0.003 kg m² ; 0.012 kg m²

3. Find the radius of gyration of a rod of mass 100 g and length 100 cm about an axis passing through its centre and perpendicular to length

Ans: 0.289 m

☞ Ring

Q.26 State an expression for M.I. of circular ring

A) about its axis through its centre and perpendicular to its plane

B) about any of its diameter.

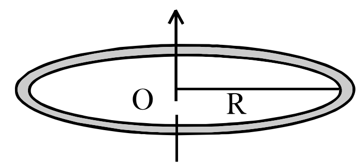
C) about its tangent in its plane

D) about its tangent perpendicular to the plane.

Ans: Let, R - radius of the ring

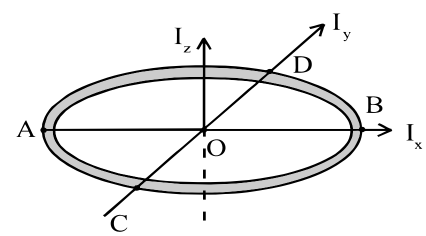
M - mass of the ring

A) The moment of inertia I of the ring about and axis through its centre and perpendicular to its plane



$$\boxed{I = MR^2}$$

B) M. I. of ring about its diameter.



i. Moment of inertia of the ring about all the diameters will be same.

$\therefore$ $I_x = I_y = I_D$

where, I_x - M.I. about X - axis.

I_y - M.I. about Y - axis.

iii. By using principle of perpendicular axes,

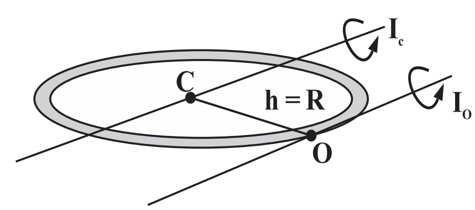
$I_z = I_x + I_y$

$\therefore$ $MR^2 = I_D + I_D = 2I_D$

$\therefore$ $$\boxed{I_D = \frac{MR^2}{2}}$$

C) About its tangent in its plane

Axis as a tangent in the plane is parallel to the axis passing about the diameter in the plane



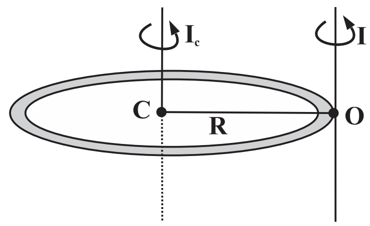
By principle of parallel axes,

$I_O = I_c + MR^2$

$$= \frac{MR^2}{2} + MR^2 = \boxed{\frac{3}{2}MR^2}$$

D) About its tangent perpendicular to the plane.

Axis as a tangent perpendicular to plane is parallel to the axis passing from centre and perpendicular to plane



By principle of parallel axis,

$$I_O = I_c + MR^2 = MR^2 + MR^2 = \boxed{2\ MR^2}$$

Type - XIV
Numerical based on Ring

Formula used

1. M.I of Ring about an axis passing from centre and perpendicular to plane.
$I = MR^2$

2. Radius of gyration
$I = MK^2$
$\therefore$ $K = \sqrt{\dfrac{I}{M}}$

3. Parallel axes theorem
$I_0 = I_c + mh_2$

4. Perpendicular axis theorem
$I_z = I_x + I_y$

1) Radius of ring is 2 cm and its mass is 20 gm. What will be is moment of inertia about an axis passing through its centre and perpendicular to its plane?

Data : R = 2cm = 2×10^{-2} m
M = 20 gm = 20×10^{-3} kg.

To find: I

Formula: M.I. of a ring about on axis passing through centre and perpendicular to it is,
$I = MR^2$

Solution : $I = MR^2$
$= 20 \times 10^{-3} \times 2 \times 10^{-2} \times 2 \times 10^{-2}$
$= 80 \times 10^{-7}$

$\therefore$ $I = 8 \times 10^{-6}$ kg - m^2

Ans : Moment of inertia of ring is 8×10^{-6} kgm^2

Problem for Practice

1. Radius of ring is 3 cm and its mass is 30gm. What will be is moment of inertia about an axis passing through its centre and perpendicular to its plane?

Ans: 27×10^{-6} kg - m^2

2. Calculate moment of inertia of a ring of mass 200 g and radius 0.5 m about an axis of rotation passing through diameter and tangent perpendicular to its plane.

Ans: 0.025 kgm^2,0.1kgm^2

3. Calculate the ratio of radii of gyration of a circular ring and a disc of the same radius about the axis passing through their centres and perpendicular to their planes.

Ans: $\sqrt{2}$:1

☞ Disc

Q.27 Derive expression for moment of ineria of a uniform disc about an axis passing through center and perpendicular to the plane.

Ans:

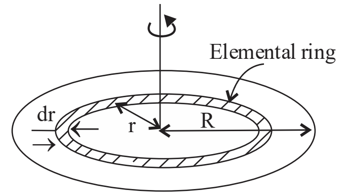
i. Consider a uniform disc of mass M and radius R rotating about its own axis, which is the line perpendicular to its plane and passing through its center.

Let surface density of disc be σ

$\therefore$ $\sigma = \dfrac{M}{A} = \dfrac{M}{\pi R^2}$

ii. As it is a uniform circular object, it can be considered to be consisting of a number of concentric rings of radii increasing from (practically) zero to R.

iii. One of such rings of mass 'dm' is shown by shaded portion in the figure.



iv. Width of this ring is 'dr', which is so small that the entire ring can be considered to be of average radius r.

v. Area of this ring, A = (2 π r)dr

$\therefore \quad \sigma = \dfrac{dm}{(2\pi r)dr}$

$\therefore \quad dm = (2\pi\sigma r)dr$

vi. As it is a ring, this entire mass is at a distance r from the axis of rotation.
Thus, the moment of interia of this ring,
$I_r = dm\,(r^2)$

vii. Moment of inertia (I) of the disc can now be obtained by integrating I_r from r = 0 to r = R.

$$\therefore \quad I = \int_0^R I_r = \int_0^R dm.r^2 = \int_0^R 2\pi\sigma r.dr.r^2$$

$$= 2\pi\sigma\int_0^R r^3.dr = 2\pi\sigma\left(\frac{R^4}{4}\right)$$

$$\therefore \quad I = 2\pi\left(\frac{M}{\pi R^2}\right)\left(\frac{R^4}{4}\right) = \frac{1}{2}MR^2$$

This is required expression for M.I of disc about an axis passing from centre an perpendicular to plane

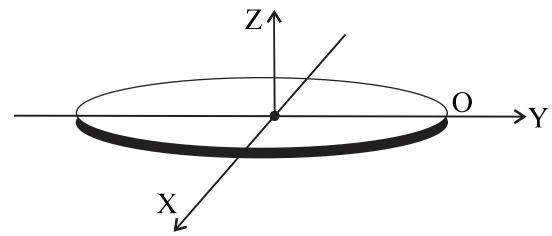
Q.28 Obtain an expression for moment of inertia of a thin uniform disc
A) about an axis coinciding with some diameter of the disc.
B) about an axis that is tangent at a point on the circumference of the disc is perpendicular to the plane of the disc.
C) about an axis that is tangent at a point on the circumference of the disc and is in the plane of the disc.

Ans:

A) About an axis coinciding with some diameter of the disc.

i. M.I. of a disc about on axis passing through centre and perpendicular to its plane (along z - axis) is,

$$I_z = I_c = \frac{MR^2}{2}$$



ii. Moment of inertia of the disc about all the diameters will be same.

$\therefore \quad I_x = I_y = I_D$

where, I_x – M.I. about X - axis.
I_y – M.I. about Y - axis.

iii. By using principle of perpendicular axes,

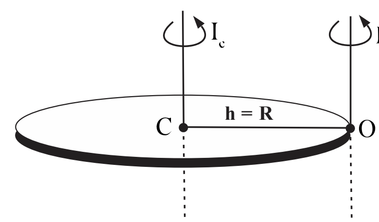
$I_z = I_x + I_y$

$\therefore \quad \dfrac{MR^2}{2} = I_D + I_D$

$\dfrac{MR^2}{2} = 2I_D$

$$\therefore \quad \boxed{I_D = \frac{MR^2}{4}}$$

B) About an axis that is tangent at a point on the circumference of the disc and perpendicular to the plane of the disc.

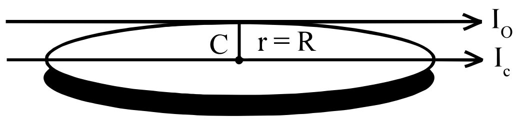


By using principle of parallel axis,
M.I. about an axis i.e. tangent at a point on the circumference perpendicular to the plane of the disc

$$I_o = I_c + Mh^2 = \frac{MR^2}{2} + MR^2$$

$$\therefore \quad \boxed{I_0 = \frac{3}{2}MR^2}$$

C) about an axis that is tangent at a point on the circumference of the disc and is in the plane of the disc.



By using principle of parallel axis,
Tangent at a point on the circumference of the disc and is in the plane of the disc

$$I_o = I_c + Mh^2 = \frac{MR^2}{4} + MR^2$$

$\therefore$ $\boxed{I_0 = \frac{5}{4} MR^2}$

Type - XV
Numerical based on Disc

Formula

1. $I = \frac{MR^2}{2}$

2. Radius of gyration
$I = MK^2$
$\therefore$ $K = \sqrt{\frac{I}{M}}$

3. Parallel axes theorem
$I_0 = I_c + Mh_2$

4. Perpendicular axis theorem
$I_z = I_x + I_y$

1) M.I. of a disc about an axis passing through its centre and perpendicular to its plane is 5 kg-m^2. Determine its M.I. about a parallel axis :
A) tangential to its rim
B) passing through a point midway between the centre and the rim.

Data : $I_c = 5 \text{kg m}^2$

To Find: i. M.I of a disc about tangent to its rim
ii. M.I of a disc about axis passing through a point midway between the centre and the rim

Formula: i. $I_0 = I_c + Mh^2$

ii. $I_c = \frac{MR^2}{2}$

Solution :

i. M.I. of the disc about an axis passing through its centre and perpendicular to its plane is given by, $I_c = \frac{MR^2}{2}$

$\therefore$ $\frac{MR^2}{2} = 5$

$\therefore$ $MR^2 = 10$

ii. M.I. of the disc about parallel axis tangential to its rim.

$\therefore$ $I_c = \frac{MR^2}{2}$ and h = R

By principle of parallel axes,
$I_0 = I_c + Mh^2$

$$I_0 = I_c + MR^2 = \frac{MR^2}{2} + MR^2$$

$\therefore$ $= \frac{3}{2} MR^2 = \frac{3}{2} \times 10 = 15 \text{ kg m}^2$

iii. M.I. of the disc about a parallel axis passing through a point mid-way between the centre and the rim.

We have $I_c = \frac{MR^2}{2}$ and $h = \frac{R}{2}$

By principle of parallel axes,
$I_0 = I_c + Mh^2$

$\therefore$ $I_0 = \frac{MR^2}{2} + M\left(\frac{R}{2}\right)^2 = \frac{MR^2}{2} + \frac{MR^2}{4}$

$$= \frac{2MR^2 + MR^2}{4} = \frac{3}{4} MR^2 = \frac{3}{4} \times 10$$

$= 7.5 \text{ kgm}^2$

Ans : The M.I. of the disc about a parallel axis tangential to its rim is 15 kg m^2 and the M.I. about the parallel axis passing through a point mid-way between the centre and the rim is 7.5 kg m^2.

Problem for Practice

1. The M.I. of a uniform circular disc about an axis passing through its centre and perpendicular to its plane is $\frac{1}{2}$ MR^2. Find the distance of a parallel axis from the centre of mass about which the M.I. of the disc is MR^2. Radius of disc = $\sqrt{18}$ cm.

Ans : 3cm

2. Radius of gyration of a disc about an axis passing through its centre and perpendicular to its plane is 4 cm. Find its radius of gyration about a diameter.

Ans: $2\sqrt{2}$ cm

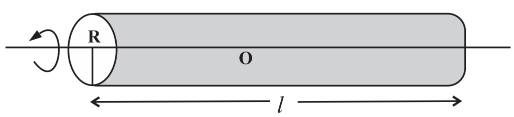
3. Find the M.I. of a thin uniform circular disc of mass 10 kg and radius 50 cm about its tangent
   a. in its plane
   b. perpendicular to the plane.

Ans: 3.125kgm^2 and 3.75kgm^2

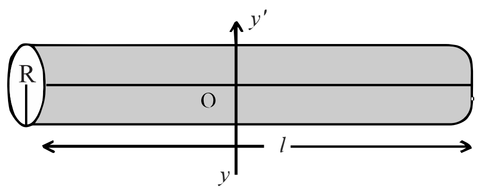
(D) Solid Cylinder

i. M.I of a solid cylinder about its own axis

$$I = \frac{1}{2}MR^2$$



ii. M. I. of solid cylinder about an axis through its centre and perpendicular to its own axis of symmetry.



$$I = \frac{MR^2}{4} + \frac{Ml^2}{12}$$

Type - XVI
Numerical based on Solid cylinder

Formula used

1. $\frac{MR^2}{2}$ (About its own axis)
2. $I = \frac{MR^2}{4} + \frac{Ml^2}{12}$ (Axis perpandicular to length)

1) A solid cylinder of uniform density of radius 1 cm has mass of 50 g. If its length is 20 cm, Calculate its moment of inertia about

i. its own axis of rotation passing through the centre,

ii. an axis passing through its centre and perpendicular to its length.

Data: $M = 50g = 50 \times 10^{-3}$ kg

$R = 1$ cm $= 1 \times 10^{-2}$ m

$l = 20$ cm $= 20 \times 10^{-2}$ m

To Find: i. M.I of cylinder about its own axis (I_c)
ii. M.I about an axis perpendicular to its length (I)

Formula:

i Moment of inertia about its own axis is

$$I_c = \frac{1}{2}MR^2$$

ii. Moment of inertia about an axis perpendicular to its length is given by

$$\therefore \quad I = \frac{1}{4}MR^2 + \frac{Ml^2}{12}$$

Solution:

i. Moment of inertia about its own axis is

$$I_c = \frac{1}{2}MR^2$$

$$\therefore \quad I_c = \frac{1}{2} \times 50 \times 10^{-3} \times (1 \times 10^{-2})^2$$

$$= \frac{1}{2} \times 50 \times 10^{-3} \times 10^{-4}$$

$$= 25 \times 10^{-7} \text{ kg m}^2$$

ii. Moment of inertia about an axis perpendicular to its length is given by

$$\therefore I_c = \frac{1}{4}MR^2 + \frac{Ml^2}{12}$$

$$= \frac{1}{4} \times 50 \times 10^{-3} \times \left(10^{-2}\right)^2 + \frac{50 \times 10^{-3} \times (20 \times 10^{-2})^2}{12}$$

$$= 12.5 \times 10^{-7} + 1666.6 \times 10^{-7}$$

$$= 1679.1 \times 10^{-7} = 1.679 \times 10^{-4} \text{ kgm}^2$$

Ans : M.I about its own axis is 25×10^{-7} kgm^2 and M.I about an axis perpendicular to its length is 1.679×10^{-4} kgm^2

2) A solid cylinder of uniform density of radius 2 cm has mass of 50 g. If its length is 12 cm, calculate its moment of inertia about an axis passing through its centre

and perpendicular to its length. (Feb. 2014)

Data : $R = 2\text{ cm} = 2 \times 10^{-2}\text{m}$
$M = 50\text{g} = 50 \times 10^{-3}\text{ kg} = 5\times10^{-2}\text{ kg}$
$l = 12\text{ cm} = 12 \times 10^{-2}\text{ m}$

To find: M.I about axis passing from centre and perpendicular to length (I)

Formula : $I = \frac{Ml^2}{12} + \frac{MR^2}{4}$

$$I = \frac{M}{4}\left(\frac{l^2}{3} + R^2\right)$$

Solution : $I = \frac{5 \times 10^{-2}}{4}\left[\frac{(12\times10^{-2})^2}{3} + (2\times10^{-2})^2\right]$

$$= \frac{5 \times 10^{-2}}{4}\left[\frac{\overset{48}{\cancel{144}}\times10^{-4}}{\underset{1}{\cancel{3}}} + 4\times10^{-4}\right]$$

$$= \frac{5 \times 10^{-2}}{\underset{1}{\cancel{4}}} \times \overset{13}{\cancel{52}}\times10^{-4}$$

$= 65\times10^{-6}\text{ kgm}^2$

Ans : Moment of inertia of Cylinder is $65\times10^{-6}\text{kgm}^2$

Problem For Practice

1. A solid cylinder has a height of 12 cm, radius of 4 cm and a mass of 100 g. Find its M.I. about its axis of symmetry and also about an axis passing through its centre and perpendicular to the axis.
Ans : $8 \times 10^{-5}\text{ kgm}^2$; $16 \times 10^{-5}\text{ kg m}^2$

2. A solid cylinder of mass 200g has radius 10cm and height 50cm. Find M.I of solid cylinder about its own axis and also about an axis passing from centre and perpendicular to its length
Ans : $10\times10^{-5}\text{ kgm}^2$; 13.3×10^{-5}J kgm^2

3. Calculate the moment of inertia of a cylinder of length 1.5 m, radius 0.05 m and density $8\times10^3\text{ kg m}^{-3}$ about the axis of the cylinder.
Ans : 0.1175 kg m²

☞ Moment of Inertia : (E) Solid Sphere

M.I. of solid sphere about an axis of rotation coinciding with its diameter.

$$I = \frac{2}{5}MR^2$$

Type - XVII
Numerical based on solid sphere

Formula

1. $I = \frac{2}{5}MR^2$

2. Radius of gyration
$I = MK^2$
$\therefore$ $K = \sqrt{\frac{I}{M}}$

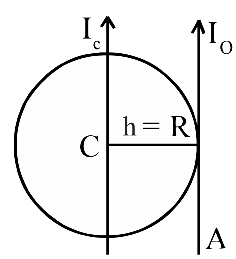
3. Parallel axes theorem
$I_0 = I_c + Mh_2$

4. Perpendicular axis theorem
$I_z = I_x + I_y$

1) Find M. I. and radius of gyration of solid sphere about a tangent to its surface.

Solution:

i. Moment of inertia of the sphere about the diameter is, $I = \frac{2}{5}MR^2$



ii. According to principle of parallel axes, M.I. about a tangent drawn to the sphere at any point,

$$I_O = I_c + Mh^2 = \frac{2}{5}MR^2 + MR^2$$

$\therefore$ $I_O = \frac{7}{5}MR^2$

iii. $MK^2 = \frac{7}{5}MR^2$

$\therefore$ $K^2 = \frac{7}{5}R^2$

$\therefore$ $K = \sqrt{\frac{7}{5}}R^2$

2) If the radius of solid sphere is doubled by keeping its mass constant, compare the moment of inertia about any diameter.

Data: $R_2 = 2R_1$

To Find: $\frac{I_1}{I_2}$

Formula: $I = \frac{2}{5} MR^2$

Solution : $I = \frac{2}{5} MR^2$

$$\therefore \quad \frac{I_1}{I_2} = \left(\frac{R_1}{R_2}\right)^2 = \left(\frac{R_1}{2R_1}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\therefore \quad \frac{I_1}{I_2} = 1:4$$

Ans :	The ratio of M.I about diameter is 1:4

★3) A big dumb-bell is prepared by using a uniform rod of mass 60 g and length 20cm Two identical solid spheres of mass 50 g and radius 10 cm each are at the two ends of the rod. Calculate moment of inertia of the dumb-bell when rotated about an axis passing through its centre and perpendicular to the length.

Data : Mass of rod, M = 60g
Length of rod, L = 20 cm
Mass of solid sphere, m = 50g
Radius of sphere, r = 10 cm

To Find : M.I of dumb - bell about an axis passing from centre and perpendicular to length

Formula :

i. M.I of rod about perpendicular bisector

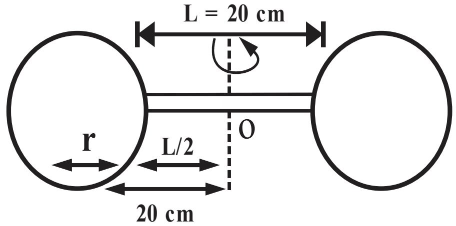
$$I = \frac{ML^2}{12}$$

ii. M.I of sphere about axis passing from centre

$$I = \frac{2}{5}MR^2$$

iii. Parallel axis theorem

$$I_0 = I_c + Mh^2$$



Solution :

i. M.I of sphere about an axis passing from point 0.

$$I_0 = I_c + Mh^2$$

$$I_0 = \frac{2}{5} Mr^2 + m\left(\frac{L}{2} + r\right)^2$$

$$= \frac{2}{5} \times \overset{10}{\cancel{50}} \times (10)^2 + 50\left[\frac{20}{2} + 10\right]^2$$

$$= 2000 + 20000 = 22000 \text{ gm cm}^2$$

ii. M.I of Rod about axis passing from point O

$$I_0 = \frac{ML^2}{12}$$

$$I_0 = \frac{\overset{5}{\cancel{60}} \times (20)^2}{\underset{1}{\cancel{12}}} = 5 \times 400 = 2000 \text{ gm cm}^2$$

iii. M.I of dumb - bell

= M. I of Rod + 2 × M. I of sphere

= 2000 + 2 (22000)

$= 2000 + 44000 = 46000 \text{ gm cm}^2$

Ans :	Moment of inertia of the dumb-bell when rotated about an axis passing through its centre and perpendicular to the length is 46000 gcm^2

4) A uniform solid sphere has a radius 0.1 m and density 6 × 10^3 kg/m^3. Find its moment of inerita about a tangent of its surface. (July 2016)

Data : R = 0.1m
$\rho = 6 \times 10^3$ kgm^3

To Find : M.I about tangent of sphere (I)

Formula : i. $I_T = I_C + MR^2$

$$= \frac{2}{5}MR^2 + MR^2$$

$$I_T = \frac{7}{5}MR^2 \quad \text{... (1)}$$

ii. $\text{Density} = \frac{\text{mass}}{\text{volume}}$

mass = Volume × density

$$M = \frac{4}{3}\pi R^3\rho \quad \text{... (2)}$$

iii. Substitute (2) in (1)

$$I_T = \frac{7}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2$$

$$I_T = \frac{28}{15}\pi R^5\rho$$

Solution : $I_T = \frac{28}{15}\pi R^5\rho$

$$I_T = \frac{28}{15} \times 3.14 \times (10^{-1})^5 \times 6 \times 10^3$$

$$= \frac{175.84}{5} \times 10^{-5} = 35.16 \times 10^{-5}\ kgm^2$$

Ans : Moment of inertia about tangent of sphere is $35.16 \times 10^{-5}\ kgm^2$

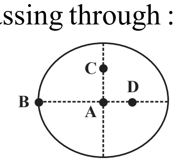
Problem for Practice

1. A uniform solid sphere has a radius 0.1 m and density $6 \times 10^3\ kg/m^3$. Find its moment of inerita about a tangent of its surface.
   Ans : 0.3517 kgm²
2. Calculate the moment of inertia of the earth about its diameter, taking it to be a sphere of 10^{25} kg and diameter 12800 km
   Ans : 1.64×10^{38} kgm²
3. The radius of a sphere is 5 cm . Calculate the radius of gyration about (i) its diameter and (ii) about any tangent.
   Ans : i. 3.16 cm , ii. 5.915 cm

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 7)

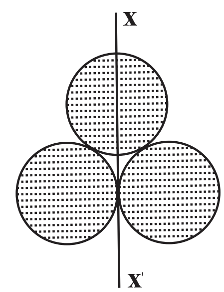
1. The ratio of radii of gyration of a circular ring and a circular disc, of the same mass and radius, about an axis passing through their centers and perpendicular to their planes are
   a. $\sqrt{2} : 1$ b. $1 : \sqrt{2}$
   c. 3 : 2 d. 2 : 1

2. The moment of inertia of a uniform circular disc is maximum about an axis perpendicular to the disc and passing through :
   a. B
   b. C
   c. D
   d. A

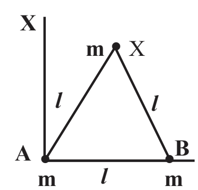


3. Four identical thin rods each of mass M and length l, form a square frame. Moment of inertia of this frame about and axis through the centre of the square and perpendicular to its plane is:
   a. $\frac{2}{3}Ml^2$ b. $\frac{13}{3}Ml^2$
   c. $\frac{1}{3}Ml^2$ d. $\frac{4}{3}Ml^2$

4. Three identical spherical shells,each of mass m and radius r are placed as shown in figure. Consider an axis XX' which is touching to two shells and passing through diameter of third shell. Moment of inertia of the system consisting of these spherical shells about XX' axis is
   a. $3mr^2$
   b. $\frac{16}{5}mr^2$
   c. 4 mr
   d. $\frac{11}{5}mr^2$



5. Three particles, each of mass m gram, are situated at the vertices of an equilateral triangle ABC of side l cm (as shown in the figure). The moment of inertia of the system about a line AX perpendicular to AB and in the plane of ABC, in gram-cm² units will be
   a. $\frac{3}{2}m\ell^2$
   b. $\frac{3}{4}m\ell^2$
   c. $2m\ell^2$ d. $\frac{5}{4}m\ell^2$



6. A composite disc is to be made using equal masses of aluminium and iron so that it has as high a moment of inertia as possible. This is possible when
   a. the surfaces of the discs are made of iron with aluminium inside
   b. the whole of aluminium is kept in the

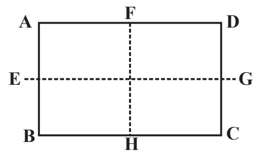
core and the iron at the outer rim of the disc

c. the whole of the iron is kept in the core and the aluminium at the outer rim of the disc

d. the whole disc is made with thin alter nate sheets of iron and aluminium

7. In a rectangle ABCD (BC = 2AB). The moment of inertia is minimum along axis through

a. BC
b. BD
c. HF
d. EG



8. Moment of inertia of a uniform circular disc about a diameter is I. Its moment of inertia about an axis $\perp$ to its plane and passing through a point on its rim will be

a. 5 I b. 3 I c. 6 I d. 4 I

9. Two copper circular discs are of the same thickness . The diameter of A is twice that of B. The moment of inertia of A as compared to that of B is (BHU 1994;MPPET 2003)

a. twice as large b. four times as large
c. 8 times as large d. 16 times as large

10. The moment of inertia of a meter scale of mass 0.6 kg about an axis perpendicular to the scale and located at the 20cm position on the scale in $kg\,m^2$ is : (Breadth of the scale is negligible)

a. 0.074 b. 0.104
c. 0.148 d. 0.208

Try Yourself

11. The moment of inertia of a uniform circular disc of radius R and mass M about an axis touching the disc at its diameter and normal to the disc is

a. $\frac{2}{5}MR^2$ b. $\frac{3}{2}MR^2$

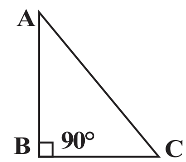
c. $\frac{1}{2}MR^2$ d. MR^2

12. The moment of inertia of a thin uniform rod of mass M of length L about an axis passing through its midpoint and perpendicular to its length is I_0. Its moment of inertia about an axis passing through one of its ends and perpendicular to its lengths is

a. $I_0+ML^2/2$ b. $I_0+ML^2/4$
c. I_0+2ML^2 d. I_0+ML^2

13. There is a flat uniform triangular plate ABC such that AB = 4 cm, BC = 3 cm and angle ABC = 90°. The moment of inertia of the plate about AB, BC and CA as axis is respectively I_1, I_2 and I_3. Which one of the following true?

a. $I_3>I_2$
b. $I_2>I_1$
c. $I_3>I_1$
d. $I_1>I_2$



14. The moment of inertia of a disc of mass M and radius R about an axis, which is tangential to the circumference of the disc and parallel to its diameter, is

a. $\frac{3}{2}MR^2$ b. $\frac{2}{3}MR^2$

c. $\frac{5}{4}MR^2$ d. $\frac{4}{5}MR^2$

15. The radius of a solid sphere is R and its density D. When it is made to rotate about an axis passing through any diameter of sphere, expression for its moment of inertia is

a. $\frac{8}{7}\pi DR^5$ b. $\frac{8}{15}\pi DR^5$

c. $\frac{28}{15}\pi DR^5$ d. $\frac{28}{5}\pi DR^5$

16. Four point size bodies each of mass M are fixed at four corners of a light squre frame of side length L.The moment of inertia of the four bodies about an axis perpendicular to the plane of frame and passing through its centre is

a. $4ML^2$ b. $2\sqrt{2}ML^2$
c. $2ML^2$ d. $\sqrt{2}ML^2$

17. Three point sized bodies each of mass M are fixed at three corners of light triangular frame of side length L. About an axis perpendicular to the plane of frame and passing through centre of frame the moment of inertia of three bodies is

a. ML^2 b. $\frac{3ML^2}{2}$

c. $\sqrt{3}ML^2$ d. $3ML^2$

18. A diatomic molecule is formed by two atoms which may be treated as mass points m_1 and m_2 joined by a massless rod of length r. Then the moment of inertia of molecule about an axis passing through centre of mass and perpendicular to the rod is :

a. zero b. $(m_1 + m_2)r^2$

c. $\left(\frac{m_1 m_2}{m_1 + m_2}\right)r^2$ d. $\left(\frac{m_1 + m_2}{m_1 m_2}\right)r^2$

19. I is moment of inertia of a thin circular plate about an axis of rotation perpendicular to the plane of plate and passsing through its centre. The moment of inertia of same plate about an axis passing through its diameter is

a . 2I b. $\sqrt{2}I$ c. $\frac{I}{\sqrt{2}}$ d. $\frac{I}{2}$

20. The moment of inertia of a thin uniform rod of mass M and length L about an axis perpendicular to the rod, through its centre is I. The moment of inertia of the rod about an axis perpendicular to the rod through its end point is (E-1999)

a. I/4 b. I/2 c. 2I d. 4I

1.8 Angular momentum

Q.25 What is angular momentum give its SI unit and dimension

i. The sum of moments of momentum of all the particles of a body rotating about a given axis is called angular momentum of the body about that axis.

ii. Symbol : L

iii. Formula:

Angular momentum =
Linear momentum × moment arm
L = p × r = mvr

iv. In terms of vector quantity
$\vec{L} = \vec{r} \times \vec{p}$
Where $\vec{r}$ is the position vector from axis of rotation

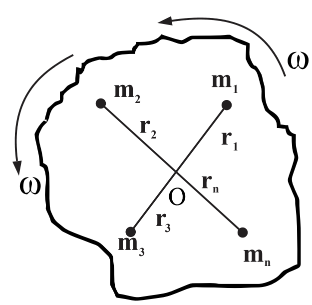
v. **SI unit :** kg m^2/s.
CGS unit : gcm^2/s.

vi. **Dimensions :** [$M^1 L^2 T^{-1}$].

Q.26 Show that Angular momentum (L) = I ω .

Ans:

i. Consider a rigid body rotating with a uniform angular velocity w about a fixed axis passing through the point O.



ii. Suppose that the body consists of n particles of masses m_1, m_2, m_3 .... m_n,
situated at distances r_1, r_2, r_3, ...r_n respectively from the given axis.

iii. As the body rotates, all the particles perform uniform circular motion with same angular velocity ω

iv. The particle of mass m_1 performs uniform circular motion with angular velocity ω and the radius r_1
Linear velocity of this first particle,
$v_1 = r_1 \omega$
Linear momentum of the first particle,
$P_1 = m_1 v_1 = m_1 r_1 \omega$
Angular momentum of the first particle,
$L_1 = (m_1 r_1 \omega) r_1 = m_1 r_1^2 \omega$,

v. Similarly, angular momentum of the second particle,
$L_2 = m_2 r_2^2 \omega$,
angular momentum of the third particle,
$L_3 = m_3 r_3^2 \omega$,

angular momentum of the n[th] particle,

$L_n = m_n r_n^2\ \omega$

vi. Now, angular momentum of rotating body is equal to the sum of the moments of momentum of all the particles of body.

$L = L_1 + L_2 + L_3 \ ..... + \ L_n$

$= m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega + ... + m_nr_n^2\omega$

$= [m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ... + m_nr_n^2]\omega$

$= \left[\sum_{i=1}^{n} m_i r_i^2\right]\omega$

But, $\sum_{i=1}^{n} m_i r_i^2 = I = \text{M.I.of a body}$

$\therefore$ $L = I\omega$

Angular momentum = M. I. × angular velocity

Key Point

1. Relation between K.E of body and angular momentum

$E_r = \frac{1}{2}I\omega^2$

$= \frac{L^2}{2I} = \frac{L\omega}{2} = \frac{MK^2\omega^2}{2} = \frac{mv^2}{2} \times \frac{K^2}{R^2}$

Type - XVIII

Numerical based on Angular momentum and K.E of rotational motion

Formula used

1. $L = I\omega$
2. $L = p \times r = mvr$
3. $\text{K.E. of rotating body} = \frac{1}{2}I\omega^2$

1) A flywheel rotating about an axis passing through its centre and at right angles to its plane loses 100J of energy when slowing down from 60 rpm to 30 rpm. Find its M.I. What is the change of its angular momentum?

Data : $\Delta E = 100$ J,

$n_1 = 60 \text{ rpm} = \frac{60}{60} = 1 \text{ Hz}$

$n_2 = 30 \text{ rpm} = \frac{30}{60} = \frac{1}{2} \text{ Hz}$

To Find : i. I ii. ΔL

Formula: i. $\Delta L = L_2 - L_1$

ii. $E = \frac{1}{2}I\omega^2$

iii. $L = I\omega$

iv. $\omega = 2\pi n$

Solution :

i. $\omega_1 = 2\pi n_1 = 2\pi \text{ rad/s}$

ii. $\omega_2 = 2\pi n_2 = 2\pi \times \frac{1}{2} = \pi \text{rad/s}$

iii. $\Delta E = (-100 \text{ J})$ (As it is loss)

$\Delta E = E_2 - E_1$

$-100 = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2 = \frac{1}{2}I\left(\omega_2^2 - \omega_1^2\right)$

$\therefore$ $-100 = \frac{1}{2}I\left[\pi^2 - (2\pi)^2\right] = -\frac{3}{2}I\pi^2$

$\therefore$ $I = \frac{100}{\pi^2} \times \frac{2}{3} = 6.762 \text{ khm}^2$

iv. $\Delta L = L_2 - L_1 = I\omega_2 - I\omega_1 = I(\omega_2 - \omega_1)$

$= \frac{200}{3\pi^2}[\pi - 2\pi] = \frac{-200}{3\pi}$

$= -21.23 \text{ kg m}^2/\text{s}$

Ans : The angular momentum decreases by 21.23 kg m2/s.

2) Determine the K.E. of a circular disc rotating with a speed of 60 r.p.m. about an axis passing through a point on its circumference and perpendicular to its plane. The circular disc has a mass of 10 kg and radius one metre.

Data : $n = 60 \text{ rpm} = \frac{60}{60} = 1 \text{ Hz}$

$\omega = 2\pi n = 2\pi \times 1 = 2\pi \text{ rad/s}$

M = 10 kg

R = 1 m

To Find : K.E of disc (E)

Formula: i. $I_c = \frac{MR^2}{2}$

ii. $I_0 = I_c + Mh^2$

iii. $E = \frac{1}{2} I\omega^2$

Solution:

i. M.I. of the disc about an axis passing through a point on its circumference and perpendicular to the plane

$I_0 = I_c + mh^2$

$= \frac{MR^2}{2} + MR^2$

$I_0 = \frac{3}{2} MR^2$

$= \frac{3}{2} \times 10 \times (1)^2$

$I_0 = 15 \text{ kgm}^2$

ii. K.E. of the disc $= \frac{1}{2} \omega^2$

$= \frac{1}{\not{2}_1} \times 15 \times \not{4}^{2} \pi^2$

$= 30 \times 10 = 300$ J

Ans :	K.E of disc is 300 J

★3) **A flywheel is a mechanical device specifically designated to efficiently store rotaional energy. For a particular machine it is in the form of a uniform 20 kg disc of diameter 50 cm, able to rotate about its own axis. Calculate its kinetic energy when rotating at 1200 rpm. (Use $\pi^2 = 10$). Calculate its moment of inertia, in case it is rotated about a tangent in its plane.**

Data: M = 20 kg,

D = 50 cm = 0.5 m,

$R = 0.25 \text{ m} = \frac{1}{4} \text{m},$

$n = 1200 \text{ rpm}, = \frac{1200}{60} = 20\text{Hz}$

$\omega = 2\pi n = 2\pi \times 20 = 40\,\pi \text{ rad/s}$

To find: i. Rotational Kinetic Energy (E)

ii. Moment of inertia about tangent to its plane (I).

Formulae: i. $K.E. = \frac{1}{2} I\omega^2$

ii. M.I. of disc about its axis as tangent in plane

Solution: $I = \frac{5}{4} MR^2$

i. $K.E. = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{MR^2}{2}\right) \times \omega^2$

$K.E. = \frac{1}{2} \times \frac{5}{8} \times (40\pi)^2 \quad .... \left(I = \frac{MR^2}{2}\right)$

$= \frac{1}{2} \times \frac{5}{4} \times 40 \times 40 \times 10 = 5000$ J

ii. M.I. about axis of disc

$I = \frac{5}{4} MR^2 = \frac{5}{4} \times 20 \times \left(\frac{1}{4}\right)^2$

$= \frac{100}{64} = 1.5625 \text{ kgm}^2$

Ans :	
i.	Rotational Kinetic energy (when it is rotating about its axis) is 5000 J.
ii.	Moment of inertia of disc about a tan gent is its plane is 1.5625 kgm^2

4) **A wheel of moment of inertia 1 kgm² is rotating at a speed of 40 rad/s. Due to friction on the axis, the wheel comes to rest in 10 mintes. Calculate the angular momentum of the wheel, two minutes before it comes to rest.**

Data: $I = 1 \text{kgm}^2$

$\omega_0 = 40$ rad/s

$\omega = 0$

t = 10 min = 600 sec

To find: L (two minutes before coming to rest)

Formula: i. $\omega = \omega_0 + \alpha t$

ii. $L = I\omega$

Solution:

i. In 10 min, wheel comes to rest,

$\therefore \quad \omega = \omega_0 + \alpha t$

$0 = 40 + \alpha \times 600$

$\therefore \quad -\alpha \times 600 = 40$

$\alpha = -\frac{40}{600} = -\frac{1}{15} \text{ rad/s}^2$

ii. Let ω' be the angular velocity two minutes before coming to rest

$\therefore$ $t' = 10 - 2 = 8$ min $= 8 \times 60 = 480$ sec

$\omega' = \omega_0 + \alpha t'$

$= 40 - \frac{1}{15} \times 480 = 40 - 32 = 8$ rad/s

iii. Angulor momentum at t' = 8 min is

$L = I\,\omega' = 1 \times 8 = 8$ kgm^2/s

Ans :	Angular momentum two minute before rest is $8 kgm^2/s$

Problem for Practice

1. A flywheel of mass 300 kg rotating about an axis passing through its centre and at right angles to its plane has its angular speed increased from 60 rev/min to 120 rev/min. If the gain in K..E. thereby is 29580 J, find the M. I. of the flywheel and its radius of gyration. What is the gain in angular momentum of the flywheel?

Ans: 500 kg m^2,1.29 m, 3140 kg m^2/s

2. The angular momentum of a body is 31.4 Js and its rate of revolution is 10 cycles per second. Calculate the moment of inertia of the body about the axis of rotation.

Ans : 0.5 kgm^2

3. A ring of diameter 0.4m and of mass 10 kg is rotating about its axis at the rate of 2100 rpm. Find (i) moment of inertia (ii) angular momentum and (iii) rotational K.R. of the ring

Ans : 9680J.

4. A disc of diameter 50 cm and mass 2 kg rotates about an axis passing through its centre and at right angles to its plane with a frequency of 8 rev per sec. Find the angular momentum of the disc and the rotational K.E.

Ans : 3.14 kg m^2/s; 78.88 J

5. When the angular velocity of a disc rotating about an axis perpendicular to its plane and passing through its centre changes from 30 rad/s to 35 rad/s, the change in angular momentum is 60 kgm^2/s. What is the change in kinetic energy of the disc? **Ans : 1950J**

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 8)

1. When a 12000 J of work is done on a flywheel, its frequency of rotation increases from 10 Hz to 20 Hz. The moment of inertia of flywheel about its axis of rotation is (Take, $\pi^2 = 10$) (2019)

a. 1 kg-m^2 b. 2 kg-m^2
c. 1.688 kg-m^2 d. 1.5 kg-m^2

2. A mass is whirled in a circular path with constant angular velocity and its linear velocity is v. If the string is now halved keeping the angular momentum same, the linear velocity is

a. 2v b. $\frac{v}{2}$ c. $\frac{v}{4}$ d. $v\sqrt{2}$

3. A wheel of moment of inertia 2 kg-m^2 is rotating about an axis passing through centre and perpendicular to its plane at a speed 60 rad/s. Due to friction, it comes to rest in 5 min. The angular momentum of the wheel three minutes beforeit stops rotating is

a. 24kg-m^2/s b. 48kg-m^2/s
c. 72 kg-m^2/s d. 96kg-m^2/s

4. A cord is wound the circumference of wheel of radius r. The axis of the wheel is horizontal and moment of inertia about it is I. The weight mg is attached to the end of the cord and falls from rest. After falling through a distance h, the angular velocity of the wheel will be

a. $[mgh]^{1/2}$ b. $\left[\frac{2mgh}{1+2mr^2}\right]^{1/2}$

c. $\left[\frac{2mgh}{1+mr^2}\right]^{1/2}$ d. $\left[\frac{mgh}{1+mr^2}\right]^{1/2}$

5. A fly wheel rotating about a fixed axis has a kinetic energy of 360 joule when its angular speed is 30 radian/sec, The moment of inertia of the wheel about the axis of rotation is

a. 0.6 kg/m^2 b. 0.15 kg m^2
c. 0.8 kg m^2 d. 0.75 kg m^2

Try Yourself

6. The moment of inertia of a body about a given axis is 1.2 kg m^2. Initially, the body is at rest. In order to produce a rotational kinetic energy of 1500 joule, an angular acceleration of 25 radian /sec^2 must be applied about that axis for a duration of

a. 4 seconds b. 2 seconds
c. 8 seconds d. 10 seconds

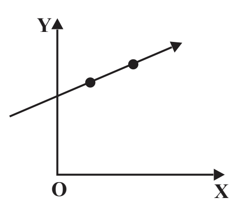
7. A ring of mass m and radius r rotates about an axis passing through its centre and perpendicular to its plane with angular velocity ω. Its kinetic energy is

a. $\frac{1}{2}mr^2\omega^2$ b. $mr\omega^2$
c. $mr^2\omega^2$ d. $\frac{1}{2}mr\omega^2$

8. When a mass is rotating in a plane about a fixed point, its angular momentum is directed along:

a. a line perpendicular to the plane of rotation
b. the line making an angle of 45° to the plane of rotation.
c. the radius
d. the tangent to the orbit

9. A particle of mass m moves in the XY plane with a velocity v along with straight line AB. If the angular momentum of the particle with respect to origin O is L_A when it is at A and L_B when it is at B, then



a. $L_A = L_B$
b. the relationship between L_A and L_B
c. $L_A < L_B$
d. $L_A > L_B$

10. Two bodies have their moments of inertia I and 2I respectively about their axis of rotation. If their kinetic energies of rotation are equal, their angular momenta will be in the ratio

a. 2 : 1 b. 1 : 2
c. $\sqrt{2}$: 1 d. 1 : $\sqrt{2}$

1.9 Torque in terms of M.I

Note:

i. *Torque is the measure of the force that can cause an object to rotate about an axis.*

ii *Force is what causes an object to accelerate in linear kinematics. Similarly, torque is what causes an angular acceleration. Hence, torque can be defined as the rotational equivalent of linear force.*

iii. ***Moment of force :***

a. *Ability of force to produce rotational motion is called as momentof force. The moment of force or the torque due to force gives turning effect of force about fixed axis.*

b. *Moment of force is measured by the product of magnitude of force and Perpendicular distance of the line of action of force from the axis of rotation.*

c. *Formula*

$$\begin{pmatrix} \text{Moment of} \\ \text{force} \end{pmatrix} = (\text{Force}) \times \begin{pmatrix} \text{Perpendicular} \\ \text{distance} \\ \text{from axis of} \\ \text{rotation} \end{pmatrix}$$

$$\tau = I\frac{d\omega}{dt} = \frac{d}{dt}I\omega = \frac{dL}{dt}$$

$\hat{u}$ is unit vector along the direction of $\vec{\tau}$

d. *Direction of torque is perpendicular to plane containing $\vec{r}$ and $\vec{r}$*

It is determined by right handed screwrule.

Moment of force is also called as Torque

e. *SI unit of torque is N-m*

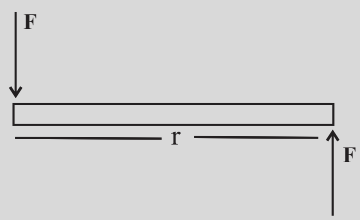
f. *Dimensions of torque are $[M^1L^2T^{-2}]$*

iv. ***Couple.***

a. Two forces equal in magnitude opposite in direction and whose line of action do not coincides with each other constitutes a couple.”

b. Couple produces purely rotational motion. Ability of couple to produce rotational motion is called as momentum of couple or torque.

c. Moment of couple is measured as product ofmagnitude of one of force and perpendicular distance between line of action of two forces.



d. Moment of couple or torque is vector. It is perpendicular to plane contains $\vec{F}$ *and* $\vec{r}$ *given by right hand screw rule.* $\vec{\tau} = \vec{r} \times \vec{F}$

e. SI unit of moment of couple is N-m.

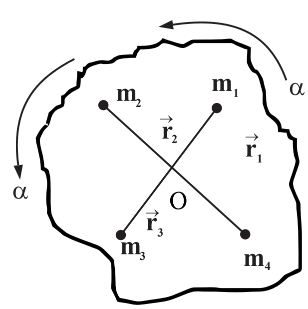
CGS unit dyne-cm.

Dimensions :$[M^1L^2T^{-2}]$

★ Q.27 Obtain the relation between the torque and angular acceleration of a rotating rigid body.

Ans:

i. Consider a rigid body rotating about a fixed axis passing through point O with uniform angular acceleration α under the action of torque τ.



ii. Suppose that the body consists of n particles of masses m_1, m_2, m_3, ... m_n, situated at distances r_1, r_2, r_3, ... r_n, from the axis of rotation respectively.

iii. As the body rotates, all the particles perform circular motion with the same angular acceleration α.

iv. The linear acceleration of the first particle, of mass m_1 is $a_1 = r_1 . \alpha$

Force acting on it $F_1 = m_1 . a_1 = m_1 r_1 . \alpha$

Moment of force acting on the first particle,

$\tau_1 = F_1 . r_1 = m_1 r_1 \alpha . r_1 = m_1 r_1^2 . \alpha$

v. Similarly, torque acting on second particle is, $\tau_2 = m_2 r_2^2 . \alpha$

torque acting on third particle is,

$\tau_3 = m_3 r_3^2 . \alpha$

torque acting on n^{th} particle is,

$\tau_4 = m_n r_n^2 . \alpha$

vi. The torque acting on the body will be the sum of the moments of forces of all the particles.

$$\tau = \tau_1 + \tau_2 + \tau_3 + ... + \tau_4$$

$$= m_1 r_1^2 . \alpha + m_2 r_2^2 . \alpha + m_3 r_3^2 . \alpha + ... + m_n r_n^2 . \alpha$$

$$= \left[\sum_{i=1}^{n} m_i r_i^2 \right] \alpha$$

But $\sum_{i=1}^{n} m_i r_i^2 = I =$ moment of inertia

$\tau = I . \alpha$

Thus, Torque = Moment of Inertia × angular acceleration

Key Point

i. Work done W = $\vec{\tau} \cdot \vec{\theta}$

Power P = $\vec{\tau} \cdot \vec{\omega}$

ii. The angular momentum of rigid body rotating about a given axis may be written as L = $I\omega = mr^2 \omega$. Torque acting on a rigid body rotating about a given axis with uniform angular acceleration a may be written as $\tau = I\alpha$.

As $\alpha = \frac{d\omega}{dt}$

$\therefore \quad \tau = I \frac{d\omega}{dt} = \frac{d}{dt} I\omega = \frac{dL}{dt}$

Type - XIX
Numerical based on torque

Formulae used
$\tau = I.\ \alpha$

1) A torque of 1500 Nm acting on a body produces an angular acceleration of 3.2 rad/s². Find M.I. of the body.

Data: $\tau = 1500Nm$
$\alpha = 3.2\ rad/s^2$
To find: M.I. of body (I)
Formula: $\tau = I\alpha$
Solution: $\tau = I\alpha$

$\therefore \quad I = \frac{\tau}{\alpha}$

$\therefore \quad I = \frac{1500}{3.2}$

$\therefore \quad I = 468.75\ kgm^2$

Ans : Moment of inertia of body is $468.75 kgm^2$

2) An automobile moves on a road with a speed of 54 kmh⁻¹. The radius of its wheels is 0.35 m. What is the average negative torque transmitted by its brakes to a wheel if the vehicle is brought to rest in 15 s ? The moment of inertia of the wheel about the axis of rotation is 3 kgm²

Data : $u = 54\ km/hr = 54 \times \frac{5}{18} = 15 m/s$

$R = 0.35\ m \quad \omega = 0\ rad/s$
$t = 15\ sec \quad I = 3\ kgm^2$

To find : Torque (τ)
Formula : i. $v = r\omega$

ii. $\alpha = \frac{\omega - \omega_0}{t}$

iii. $\tau = I\alpha$

Solution :

i. Intial angular velocity

$\omega_0 = \frac{u}{R} = \frac{15}{0.35}\ rad\ s^{-1}$

ii. Average angular acceleration,

$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - \frac{15}{0.35}}{15} = -\frac{1}{0.35}\ rad\ s^{-2}$

iii. Average torque transmitted by the brakes

$\tau = I\ .\ \alpha = -3 \times \frac{1}{0.35} = -8.57 kgm^2 s^{-2}$

Ans : Negative Torque transmitted to wheel is $8.75\ kgm^2 s^{-2}$

3) The speed of rotation of the body increases from 60 rpm to 90 rpm in 1 minute, Calculate the torque acting on the body, if its moment of inertia 500 kg m².

Data : $I = 500\ kg\ m^2$, $n_1 = 60\ rpm = 1\ rps.$
$n_2 = 90\ rpm = 1.5\ rps.$
$t = 1\ min = 60\ sec.$

To Find : Torque (τ)
Formula: Torque acting on body is,

$\tau = I\alpha = I\left[\frac{\omega_2 - \omega_1}{\tau}\right]$

$= I \times \frac{(2\pi n_2 - 2\pi n_1)}{\tau} = I \times 2\pi\left(\frac{n_2 - n_1}{\tau}\right)$

Solution : $\tau = 500 \times 2 \times 3.14 \left[\frac{1.5 - 1}{60}\right]$

$\tau = \frac{500 \times 2 \times 3.14 \times 0.5}{60}$

$\therefore \quad \tau = 26.17\ Nm$

Ans : Torque acting on body is 26.17 Nm

4) A circular disc of mass 10 kg and radius 0.2 m is set into rotation about an axis passing through its centre and perpendicular to its plane by applying torque of 10 Nm. Calculate the angular velocity of the disc at the end of 6 second from the start.

Data : $M = 10\ kg$, $R = 0.2\ m$, $\tau = 10\ Nm$,

$\omega_0 = 0\ rad/s \quad t = 6\ sec.$

To Find : ω

Formula : $\tau = I\alpha$, $I = \frac{MR^2}{2}$, $\omega = \omega_0 + \alpha t$

Solution :

i. $\alpha = \frac{\tau}{I} = \frac{2\tau}{MR^2} = \frac{2 \times 10}{10 \times (0.2)^2}$

$$= \frac{2\times10}{10\times0.04} = \frac{1}{0.02} = 50 \text{ rad/s}^2$$

ii. $\omega = \omega_0 + \alpha t$

$= 0 + 50 \times 6 = 300$ rad/s.

Ans : Angular velocity of disc is 300 rad/s

★5) A ceiling fan having moment of inertia 2 kg-m² attains its maximum frequency of 60 rpm in '2π' seconds. Calculate is power rating.

Data: $I = 2\text{ kgm}^2,\ \omega_0 = 0\text{ rad/s},$

$\omega = 2\pi n = 2\pi\times2 = 4\pi\text{ rad/s}, t = 2\pi s$

To find: Power (P)

Formulae: i. $\alpha = \frac{\omega-\omega_0}{t}$

ii. $\tau = I\alpha$

iii. $P = \tau\omega$

Solution:

i. $\alpha = \frac{\omega-\omega_0}{t}$

$\alpha = \frac{4\pi-0}{2\pi} = 2\text{rads}^{-2}$

ii. $\tau = I\alpha = 2\times2 = 4$ Nm

iii $p = \tau\omega = 4\times4\pi = 16\pi = 50.24$ W

Ans : The power rating of ceiling fan is 50.24 W.

Problem for Practice

1. A cylinder of length 20 cm and radius 10 cm is rotating about its central axis at an angular speed of 100 rad/s. What tangential force will stop the cylinder at a uniform rate in 10 seconds? The moment of inertia of the cylinder about its axis of rotation is 0.8 kg m²

Ans : 80 N

2. A wheel of moment of inertia 1 kg m² is rotating at a speed of 30 rad/s. Due to friction on the axis, it comes to rest in 10 minutes. Calculate the average torque of the friction.

Ans : - 450 J

3. A solid sphere of diameter 25 cm and mass 25 kg rotates about an axis through its centre. Calculate its moment of inertia. If its angular velocity changes from 2 rad/s to 22 rad/s in 5 seconds. Calculate the torque applied.

Ans : 0.39 Nm

4. A body starts rotating from rest. Due to a couple of 20 Nm it completes 60 revolutons in one minute. Find the moment of inertia of the body.

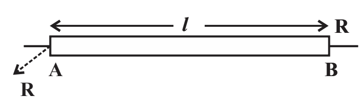
Ans : 95.48 kgm²

5. A wheel of M.I. 3 kg m² rotating at 60 rad/s is brought to rest in 5 minutes due to friction. Find the work done and the average frictional torque acting. Also find the angular momentum of the disc 2.5 minutes before the wheel stops rotating.

Ans : 0.6 Nm; 5400 J; 90 kg m²/s

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 9)

1. A rod l m long is acted upon by a couple as shown in the figure. The moment of couple is τ Nm. If the force at each end of the rod, then magnitude of each force is (sin30° = cos60°=0.5)



a. $\frac{\tau}{I}$ b. $\frac{I}{2\tau}$ c. $\frac{2\tau}{I}$ d. $\frac{2I}{\tau}$

2. Find the torque about the origin when a force of $3\hat{j}$ N act on a particle whose position vector is $2\hat{k}$ m.

a. $6\hat{j}$ b. $-6\hat{i}$ Nm

c. $6\hat{k}$ Nm d. $6\hat{i}$ Nm

3. A solid cylinder of mass 2 kg and radius 4 cm is rotating about its axis at the rate of 3 rpm. The torque required to stop after 2π revolution is:

a. 2×10^{-6} Nm b. 2×10^{-3} Nm

c. 2×10^{-4} Nm d. 2×10^{6} Nm

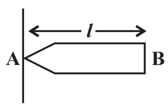
4. A uniform Circular disc of radius 50 cm at rest is free to turn about an axis which is perpendicular to its plane and passes through its centre. It is subjected to a torque which produces a constant angular acceleration of 2.0 rad s^{-2} . Its net acceleration in ms^{-2} at the end of 2.0s is approximately.

a 8.0 b. 7.0 c. 6.0 d. 3.0

5. The instantaneous angular position of a point on a rotating wheel is given by the equation $\theta(t) = 2t^3 - 6t^2$. The torque on the wheel becomes zero at.

a. t = 1s b. t = 0.5s
c. t = 0.25s d. t = 2s

6. A uniform rod AB of length l, and mass m is free to rotate about point A. The rod is released from rest in the horizontal position. Given that the moment of inertia of the rod about A is $\frac{m\ell^2}{3}$, the initial angular acceleration of the rod will be



a. $\frac{mg\ell}{2}$ b. $\frac{3}{2}g\ell$
c. $\frac{3g}{2\ell}$ c. $\frac{2g}{3\ell}$

7. A wheel having moment of inertia 2 kg-m^2 about its vertical axis, rotates at the rate of 60 rpm about this axis. The torque which can stop the wheel's rotation in one minute would be

a. $\frac{\pi}{18}N-m$ b. $\frac{2\pi}{15}N-m$
c. $\frac{\pi}{12}N-m$ d. $\frac{\pi}{15}N-m$

8. A constant torque of 1000 N-m turns a wheel of moment of inertia 200 kg-m^2 about an axis through its centre. Its angular velocity after 3 seconds is

a. 1 rad/s b. 5 rad/s
c. 10 rad/s d. 15 rad/s

9. An electric motor exerts a constant torque 5Nm on a fly wheel by which it is rotated at the rate of 420 rpm. The power of motor is

a. 110 watt b. 150 watt
c. 220 watt d. 300 watt

Try Yourself

10. A rope is wound around a hallow cylinder of mass 3kg and radius 40cm. What is the angular acceleration of the cylinder if the rope is pulled with force of 30 N?

a. 0.25 rad/s^2 b. 25 rad/s^2
c. 5 m/s^2 d. 25 m/s^2

11. Given that, $\vec{r} = 2\hat{i} + 3\hat{j}$ and $\vec{F} = 2\hat{i} + 6\hat{k}$. The magnitude of torque will be-

a. $\sqrt{405}$ N.m b. $\sqrt{410}$ N.m
c. $\sqrt{504}$ N.m d. $\sqrt{510}$ N.m

12. An automobile moves on a road with a speed of 54 km h^{-1}. The radius of its wheels is 0.45 m and the moment of intertia of the wheel about its axis of rotation is 3 kg m^2. If the vehicle is brought to rest in 15s, the magnitude of average torque transmitted by its brakes to the wheel is :

a. 8.58 kg m^2 s^{-2} b. 10.86 kg m^2 s^{-2}
c. 2.86 kg m^2 s^{-2} d. 6.66 kg m^2 s^{-2}

13. If $\vec{F}$ is the force acting on a particle having position vector $\vec{r}$ and $\vec{\tau}$ be the torque of this force about the original, then:

a. $\vec{r} \cdot \vec{\tau} > 0$ and $\vec{F} \cdot \vec{\tau} < 0$
b. $\vec{r} \cdot \vec{\tau} = 0$ and $\vec{F} \cdot \vec{\tau} = 0$
c. $\vec{r} \cdot \vec{\tau} = 0$ and $\vec{F} \cdot \vec{\tau} \neq 0$
d. $\vec{r} \cdot \vec{\tau} \neq 0$ and $\vec{F} \cdot \vec{\tau} = 0$

14. A torque of 50 Nm acts on a rotating body for 5 s. Its angular momentum is

a. increases by 250 kg m^2/s
b. increases by 10 kg m^2/s
c. increases by 55 kg m^2/s
d. decreases by 250 kg m^2/s247.

15. A constant torque acting on a uniform circular disc changes its angular momentum from L to 4L/3 in 2 seconds. Then the magnitude of the torque applied is

a. L/3 b. 2L/3
c. 3L/2 d. L/6

1.10 Law of conservation of angular momentum

★ **Q.27 State and prove the principle of conservation of angular momentum.**

Ans:

Principle of conservation of angular momentum:

If no external torque acts on a rotating body then angular momentum of the body remains constant.

Proof : We know that, $\vec{L} = \vec{r} \times \vec{p}$

Differentiating with respect to time, we get

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right)$$

$\therefore$ $$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \vec{p} \times \frac{d\vec{r}}{dt}$$

But, $\frac{d\vec{r}}{dt} = \vec{v}$, $\frac{d\vec{p}}{dt} = \vec{F}$ and $\vec{p} = m\vec{v}$

$\therefore$ above equation becomes,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + m\left(\vec{v} \times \vec{v}\right)$$

$\therefore$ $$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + 0 \qquad \left(\because \vec{v} \times \vec{v} = 0\right)$$

But, $\vec{r} \times \vec{F} = \vec{\tau}$

$\therefore$ $$\frac{d\vec{L}}{dt} = \vec{\tau}$$

Now if resultant torque (t) acting on the particle is zero, the $\frac{d\vec{L}}{dt} = 0$

$\therefore$ $\vec{L}$ is constant

Angular momentum of the system remains constant or conserved.

28. Give examples of conservation of angular momentum. OR

★ **Give suitable illustration of law of conservation of angular momentum**

Ans:

i. The speed of rotation of a person with some weights in his hands an standing on a rotating turntable increase, as he draws the weights to his chest.

ii. An ice-skater or a ballet-dancer can increase her angular velocity by folding her arms and brining the stretched leg close to the other leg.

iii. The speed of the inner layers of the whirlwind about its axis in a tornado is alarmingly high.

iv. A diver jumping from the spring board sometimes exhibits sommersaults in air before touching the water surface.

v. The angular velocity of a planet revolving in an elliptical orbit around the sun increases

Key Points

i. According to law of conservation of angular momentum

$L = I\omega$ = constant

$\therefore$ $I_1\omega_1 = I_2\omega_2$

$\therefore$ $I_1n_1 = I_2n_2$

$\therefore$ $$\frac{I_1}{T_1} = \frac{I_2}{T_2}$$

ii. If two rigid bodies with angular velocities ω_1 and ω_2 are coupled and ω is the final angular velocity of the arrangement,

$(I_1\omega_1 + I_2\omega_2) = (I_1 + I_2)\omega$

$\therefore$ $(I_1 n_1 + I_2n_1) = (I_1 + I_2) n$

iii. If the earth suddenly contracts, its moment of inertia decreases and angular velocity increases. As a result duration of the day would decrease.

iv. If radius of the earth shrinks to (l/n)th of the present radius without change in its mass, duration of the new day is $(24/n^2)$ hours.

v. If the radius of the earth shrinks to (l/n)th of the present radius without any change in the mean density, duration of the new day is $(24/n^5)$ hours.

vi. If radius of the earth 'R' shrinks at the rate of $\left(\frac{dR}{dt}\right)$ with time, variation of the new period of rotation is $\frac{dT}{dt} = 2\frac{T}{R}\left(\frac{dR}{dt}\right)$ (if mass of the earth is constant),

$$\frac{dT}{dt} = 5\frac{T}{R}\left(\frac{dR}{dt}\right)$$ (if mean density of the earth is constant).

Type - XX
Numerical based on Law of conservation of angular momentum

Formulae used

$L = I\omega$ = constant

$\therefore \quad I_1\omega_1 = I_2\omega_2$

★1) **A spherical water balloon is revolving at 60 rpm. In the course of time, 48.8% of its water leaks out. With what freaquency will the remaining balloon revolve now? Neglect all non-conservative forces.**

Data: $n_1 = 60 \text{ rpm} = \frac{60}{60} \text{ rps} = 1 \text{ rps}$

$$m_2 = \left(1 - \frac{48.8}{100}\right) m_1 = 0.512\ m_1$$

To find: Final frequency (n_2)

Formulae: i. Density$(\rho) = \dfrac{\text{mass(m)}}{\text{volume(V)}}$

ii. $V = \frac{4}{3}\pi R^3$ iii. $I = \frac{2}{5}mR^2$

iv. $I_1\omega_1 = I_2\omega_2$ v. $\omega = 2\pi n$

Solution.:

i. m = rV

$$\frac{m_1}{m_2} = \frac{\rho V_1}{\rho V_2} = \frac{V_1}{V_2} = \frac{\frac{4}{3}\pi R_1^3}{\frac{4}{3}\pi R_2^3} = \left(\frac{R_1}{R_2}\right)^3$$

$$\therefore \frac{R_1}{R_2} = \left(\frac{m_1}{m_2}\right)^{\frac{1}{3}} = \left(\frac{1000}{512}\right)^{\frac{1}{3}} = \frac{10}{8} = 1.25$$

ii. $I_1\omega_1 = I_2\omega_2$

$\therefore \quad I_1(2\pi n_1) = I_2(2\pi n_2) \quad .... \because (\omega = 2\pi n)$

$$\therefore \quad n_2 = \left(\frac{I_1}{I_2}\right) n_1 = \left(\frac{\frac{2}{5}m_1R_1^2}{\frac{2}{5}m_2R_2^2}\right) n_1 = \left(\frac{R_1^3}{R_2^3}\frac{R_1^2}{R_2^2}\right) n_1$$

$$\therefore \quad n_2 = \left(\frac{R_1}{R_2}\right)^5 n_1$$

$n_2 = (1.25)^5 \times 1 = 3.05$ rps.

Ans :	Final frequency after water leaked out is 3.05 rps.

★2) **A flywheel used to prepare earthenware pots is set into rotation at 100 rpm. It is in the form of a disc of mass 10 kg and radius 0.4m. A lump of clay (to be taken equivalent to a particle) of mass 1.6 kg falls on it and adheres to it at a certain distance x from the centre. Calculate x if the wheel now rotates at 80rpm.**

Data: $n_1 = 100 \text{ rpm} = \frac{100}{60}\text{Hz} = \frac{10}{6}\text{Hz}$,

$n_2 = 80 \text{ rpm} = \frac{80}{60}\text{Hz} = \frac{8}{6}\text{Hz}$

$\omega_1 = 2\pi n_1 = 2\pi \times \frac{10}{6} = \frac{10\pi}{3}$ rad / s

$\omega_2 = 2\pi n_2 = 2\pi \times \frac{8}{6} = \frac{8\pi}{3}$ rad / s

M = 10 kg, R = 0.4 m, m = 1.6 kg

To find: Distance from the centre of fly wheel where the lump of clay falls(x).

Formula: i. According to law of conservation of angular momentum $I_1\omega_1 = I_2\omega_2$

ii. M.I of disc

$$I = \frac{MR^2}{2}$$

Solution: $I_1\omega_1 = I_2\omega_2$

$$\therefore \left(\frac{MR^2}{2}\right)(\omega_1) = \left(\frac{MR^2}{2} + mx^2\right)\omega_2$$

$$\therefore \left[\frac{{}^{5}\cancel{10} \times (0.4)^2}{\cancel{2}}\right] \times \left(\frac{10\pi}{3}\right)$$

$$= \left(\frac{{}^{5}\cancel{10} \times (0.4)^2}{\cancel{2}} + 1.6 \times x^2\right) \times \left(\frac{8\pi}{3}\right)$$

$\therefore \quad 0.8 \times 10.5 = (0.8 + 1.6\,x^2)\, 8.4$

$\therefore \quad 8.4 = 6.72 + 13.44\,x^2$

$\therefore \quad x = \sqrt{\dfrac{1.68}{13.44}} = 0.35\text{m}$

Ans : The distance of lamp of clay from centre of disc is 0.35 m

3) A ballet dancer spins about a vertical axis at 90 rpm with arms outstretched. With the arms folded, the moment of inertia about the same axis of rotation changes to 75 % Calculated the new speed of rotation.

Data: $n_1 = 90$ r.p.m,

I_1 = M.I. with arms out stretched,

$I_2 = 0.75 I_1$ = M.I. with arms folded

To find : n_2

Formula: $I_1\omega_1 = I_2\omega_2$

Solution: $I_1\omega_1 = I_2\omega_2$

$\therefore \quad I_1(2\pi n_1) = I_2(2\pi n_2)$

$\therefore \quad n_2 = \left(\dfrac{I_1}{I_2}\right) n_1 = \dfrac{I_1}{0.75 I_1} \times 90 = \dfrac{100}{75} \times 90$

$\therefore \quad n_2 = 120$ r.p.m.

Ans : The new speed of rotation of ballet dancer is 120 r.p.m

4) What will be the duration of the day if the earth suddenly shrinks to $\frac{1}{27}$ th of its original volume, mass being unchanged?

Data: final volume = $\dfrac{1}{27}$ × initial, volume

T_1 = 24 hours

To Find : New duration of day (T_2)

Formula: $I_1\omega_1 = I_2\omega_2$

Solution :

i. final volume = $\dfrac{1}{27}$ × initial, volume

$$\frac{4}{3}\pi R_2^3 = \frac{1}{27} \times \frac{4}{3}\pi R_1^3$$

$\therefore \quad R_2^3 = \dfrac{1}{27} = R_1^3$

$\therefore \quad R_2 = \dfrac{1}{3} R_1$

ii. Using conservation of angular momentum

$I_1\omega_1 = I_2\omega_2$

$$\frac{2}{5} MR_1^2 \frac{2\pi}{T_1} = \frac{2}{5} MR_2^2 \frac{2\pi}{T_2}$$

$\therefore \quad \dfrac{R_1^2}{T_1} = \dfrac{R_2^2}{T_2}$

$$T_2 = \left(\frac{R_2}{R_1}\right)^2 T_1 = \left(\frac{1}{3}\right)^2 T_1$$

$$= \frac{T_1}{9} = \frac{24}{9} \text{ hrs} = 2.66 \text{ hrs.}$$

Ans : The new duration of day will be 2.66 hours

5) Two wheels of moment of inertia 4 kgm 2 rotate side by side at the rate of 120 rev/ min and 240 rev/min respectively in the opposite directions. If now both the wheels are coupled by means of weightless shaft so that the both the wheels now rotates with a common angular speed. Find the new speed of rotation.

Data : $I_1 = I_2 = I = 4$ kg-m^2,

$n_1 = 120$ r.p.m, $n_2 = 240$ r.p.m.

To find: New speed of rotation (N)

Formula: By the principle conservation of angular momentum,

$I\omega$ = constant

Solution : By the principle conservation of angular momentum, Iw = constant

$\therefore \quad I_2\omega_2 - I_1\omega_1 = (I_1 + I_2)\,\omega$

$I(\omega_2 - \omega_1) = 2I\omega$

$\therefore \quad 2 \times 2\pi N = 2\pi\,(n_2 - n_1)$

$\therefore \quad 2N = n_2 - n_1$

$\therefore \quad N = \dfrac{n_2 - n_1}{2} = \dfrac{240 - 120}{2} = 60$ r.p.m.

Ans : The new speed of rotation is 60 r.p.m

Problem for Practice

1. A wheel is rotating at the rate of 500 rpm on a shaft. Second identical wheel, initially at rest, is suddenly coupled to the same shaft. What is the angular speed of the resultant combination ? Assume M.I. of shaft to be negligible.

Ans: $\frac{25\pi}{3}$ **rad / s**

2. The sun rotates round itself in 27 days. What will be period of revolutionif the sun were expand to twice its prsent radius.

Ans: 108 days

3. A boy is standing on a rotating platform with his arms fully outstretched. The platform rotates with a speed of 81 r.p.m. On bringing arms close to his body, the radius of 1 gyration of the system is reduced by 10%. What is the increase in the speed of rotation?

Ans: Increase 9 r.p.m.

4. IF the earth suddenly contract by one fourth of its present radius, by how much would the day be reduced.

Ans: 10.5 hrs

5. A disc is rotating in a horizontal plane about a vertical axis at the rate of $\frac{5\pi}{3}$ rad/s. A blob of wax of mass 0.02 kg falls vertically on the disc and adheres to it at a distance of 0.05 m from the axis of rotation. If the speed of rotation thereby becomes 40 rev/min, calculate the M. I. of the disc.

Ans: 2×10^{-4} kg m^2

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 10)

1. A solid homogeneous sphere of mass M and radius R is moving on a rough horizontal surface, partly rolling and partly sliding. During this kind of motion of the sphere
   a. total kinetic energy is conserved
   b. the angular momentum of the sphere about the point of contact with the plan is conserved
   c. only the rotational kinetic energy a b o u t the centre of mass is conserved
   d. angular momentum about the centre of mass is conserved.

2. A solid sphere is rotating freely about its symmetry axis in free space. The radius of the sphere is increased keeping its mass same. Which of the following physical quantities would remain constant for the sphere?
   a. Angular velocity
   b. Moment of inertia
   c. Angular momentum
   d. Rotational kinetic energy

3. Two discs are rotating about their axes, normal to the discs and passing through the centre of the discs. Disc D^1 gas 2kg mass and 0.2 m radius and initial angular elocity of 50 rad s^{-1}. Disc D^2 has 4kg mass, 0.1 m radius and initial angular velocity of 200 rad s^{-1} The two discs are brought in contact face to face, with their axes of rotation coincident. The final angular velocity (in rad s^{-1}) of the system is:
   a. 40 b. 60 c. 100 d. 120

4. A round disc of moment of inertia I_2 about its axis perpendicular to its plane and passing through its centre is placed over another disc of moment of inertia I_1 rotating with an angular velocity ω about the same axis. The final angular velocity of the combination of discs is
   a. $\frac{(I_1+I_2)\omega}{I_1}$ b. $\frac{I_2\omega}{I_1+I_2}$
   c. ω d. $\frac{I_1\omega}{I_1+I_2}$

5. A thin circular ring of mass M ang radius r is rotating about its axis with a constant angular velocity ω. Four objects each of mass m, are kept gently to the opposite ends of two perpendicular diameters of the ring . The angular velocity of the ring will be
   a. $\frac{(M-4m)\omega}{M+4m}$ b. $\frac{M\omega}{4m}$

c. $\frac{M\omega}{M+4m}$ d. $\frac{(M+4m)\omega}{M}$

6. If the Earth shrinks such that its density becomes 8 times to the present value, then the new duration of the day in hours will be.

a. 24 b. 12 c. 6 d. 3

7. A thin uniform circular disc of mass M and radius R is rotating in a horizontal plane about an axis passing through its centre and perpendicular to the plane with an angular velocity ω. Another disc of same mass but half the radius is gently placed over it coaxially. The angular speed of the composite disc will be.

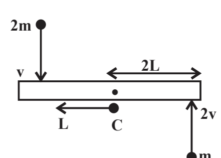
a. $\frac{5}{4}\omega$ b. $\frac{4}{5}\omega$ c. $\frac{2}{5}\omega$ d. $\frac{5}{2}\omega$

8. A ballot dancer is rotating about his own vertical axis at an angular velocity 100 revolution/minute on smooth horizontal floor. The ballot dancer folds himself close to his axis of rotation by which his moment of inertia decreases to half of initial moment of inertia then his final angular velocity is

a. 50 rpm b. 100 rpm

c. 150 rpm d. 200 rpm

9. A uniform rod of length 6L and mass 8 m is pivoted at its centre C.Two masses m and 2m with speed 2v, v as shown strikes the rod and stick to the rod. Initially the rod is at rest. Due to impact, if it rotates with angular veloity, ω then ω will be



a. $\frac{v}{5L}$

b. zero

c. $\frac{8v}{6L}$ d. $\frac{11v}{3L}$

10. The moment of inertia of a ring about an axis passing through the centre and perpendicular to its plane is l. it is rotating with angular velocity ω. Another identical ring is gently placed on it, so that their centre coincide. If both the rings are rotating about the same axis, then loss in kinetic energy is

a. $\frac{l\omega^2}{2}$ b. $\frac{l\omega^2}{4}$ c. $\frac{l\omega^2}{6}$ d. $\frac{l\omega^2}{8}$

Try yourself

11. A disc is rotating with angular velocity ω. If a child sits on it, what is conserved?

a. Linear momentum

b. Angular momentum

c. Kinetic energy

d. Moment of inertia

12. If the earth were to suddenly contract to 1/nth of its present radius without any change in its mass, the duration of the new day will be nearly

a. 24/n hours b. 24n hours

c. 24/n2 hours d. 24 n^2 hours

13. A boy suddenly comes and sits on a circular rotating table. What will remain conserved?\

a. Angular velocity

b. Angular momentum

c. Linear momentum

d. Kinetic energy

14. Two discs of moment of interia I_1, I_2 are rotating with angular velocities ω_1, ω_2 along collinear axes passing through their centre of mass and perpendicular to their plane. If the two are made to rotate combinedly along the same axis the rotational KE of system will be

a. $\frac{(I_1\omega_1+I_2\omega_2)}{2(I_1+I_2)^2}$ b. $\frac{(I_1+I_2)(\omega_1+\omega_2)^2}{2}$

c. $\frac{(I_1\omega_1+I_2\omega_2)^2}{2(I_1+I_2)}$ d. $\frac{(I_1\omega_1+I_2\omega_2)^2}{(I_1+I_2)}$

15. A ballet dancer spins about a vertical axis at 180 rpm speed with her arms outstretched. When she folds her arms MI about the same axis decreases by 40%. Her new rpm speed will be

a. 180 rpm b. 120 rpm

c. 240 rpm d. 300 rpm

16. A circular horizontal platform is rotating with angular velocity 110 radian/sec about a vertical axis passing through its centre. It has MI of 10^{-2} kg.m^2 about the axis. A blob of wax of mass 100 g is gently dropped on it and sticks to it at a distance of 10 cm from the centre. Assuming the blob to be a point mass, the angular velocity will be —

a. 50 rad/s b. 100 rad/s
c. 80 rad/s d. 120 rad/s

17. A wheel is rotating with an angular velocity of 500 rpm on a shaft. Another identical wheel initially at rest is suddenly coupled on the same shaft. Assuming MI of the shaft to be negligible, the new speed is —

a. 1000 rpm b. 250 rpm
c. 500 rpm d. 750 rpm

18. Two identical wheels are rotating about two axes passing through their centres and perpendicular to their planes. Their angular speeds respectively are 100 rpm and 50- rpm. When they are suddenly coupled by a belt of negligible MI, the new speed will be-

a. 100 rpm b. 50 rpm
c. 150 rpm d. 75 rpm

19. A disc of mass 2 kg and radius 0.2 m is rotating about an axis passing through its centre and perpendicular to its plane with an angular velocity 50 rad/s. Another disc of mass 4 kg and radius 0.1 m rotates about an axis passing through its centre and perpendicular to its plane. If the two disc are coaxially coupled, then the angular velocity of the coupled system would be

a. 150 rad/s b. 120 rad/s
c. 100/3 rad/s d. 200/3 rad/s

20. Mass remaining constant, if the earth suddenly contracts to one third of its present radius, the length of the day would be shorted by

a. 8/3 h b. 12 h c. 8 h d. 64/3 h

1.11 Rolling motion

Q.29 A body rolls without slipping, find its kinetic energy. [Mar. 13]
OR
Discuss the interlink between transtional, rotational and total kinetic energy of rolling body that rolls without slipping

Ans:

i. In case of rolling motion, body undergoes circular and rotational motion simultaneous.

ii. Total K.E. of rolling body without slipping is equal to the sum of translational K.E. and rotational K.E.

$$\begin{pmatrix}\text{Rolling}\\ \text{K.E.}\end{pmatrix} = \begin{pmatrix}\text{Translational}\\ \text{K.E.}\end{pmatrix} + \begin{pmatrix}\text{Rotational}\\ \text{K.E.}\end{pmatrix}$$

$$\text{K.E. total} = \text{K. E.}_{(T)} + \text{K. E.}_{(R)} \quad ...(1)$$

iii. Since the body does not slip, the linear velocity of the contact point at the circumference of the body must be equal to its translational velocity, say v.

$$\text{K.E.}_{(T)} = \frac{1}{2}Mv^2 \quad ...(2)$$

Where, M – mass of the body
v – velocity of the body

iv. The rotational K.E. $= \frac{1}{2}I\omega^2$,

where, $I = MK^2$ – the moment of inertia,

$\omega = \frac{v}{r}$ – the angular velocity.

$\therefore$ $$\text{K.E.}_{(R)} = \frac{1}{2}MK^2 \times \left(\frac{v}{r}\right)^2 \quad ...(3)$$

iv. Total K.E. of rolling body is,

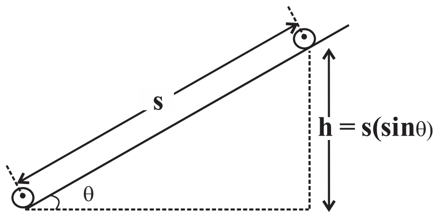
$\therefore$ equation (1) become

$$\text{K.E.}_{(total)} = \frac{1}{2}Mv^2 + \frac{1}{2}Mk^2\frac{v^2}{r^2}$$

$$= \frac{1}{2}Mv^2\left(1+\frac{k^2}{r^2}\right)$$

Q.30 A rigid body is rolling down an inclined plane. Derive expression for the acceleration along the track and speed after falling through a certain vertical distance

Ans:



i. Consider a rigid body of mass M and radius 'R' rolling down inclined plane of inclination 'q' from height 'h'.

ii. Body start from rest and rolls down, let v be linear speed acquired by the body when it reaches the bottom of inclined plane.

iii. Lost in P.E. = Gain in K. E.

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} Iw^2$$

$$= \frac{1}{2} mv^2 \left[1+\frac{K^2}{R^2}\right]$$

$$v^2 = \frac{2gh}{\left[1+\frac{K^2}{R^2}\right]} \qquad v = \sqrt{\frac{2gh}{\left[1+\frac{K^2}{R^2}\right]}}$$

iv. According to 1st kinematical equation

$v^2 = u^2 + 2as$

where, v = final velocity

u = initial velocity

s = length of plane

v. As body starts from rest, u = 0

$$a = \frac{v^2}{2s}$$

$$a = \frac{2gh}{\left(1+\frac{k^2}{R^2}\right)} \cdot \frac{1}{2s} \qquad ...(1)$$

vi. $\sin q = \frac{h}{s} \qquad s = \frac{h}{\sin\theta}$

vii. Substitute s in equation (1)

$$a = \frac{2gh}{\left(1+\frac{k^2}{R^2}\right)} \times \frac{1}{2\left(\frac{h}{\sin\theta}\right)}$$

$$a = \frac{g\sin\theta}{1+k^2/R^2}$$

This is required equation

Key Point

i. When the angular speed of all the particles of the rolling body is same, it is called rolling without slipping.

ii. Time taken in reaching the bottom is

$$t = \sqrt{\frac{2l(1+K^2/R^2)}{g\sin\theta}}$$

ii. **Body sliding down on an inclined plane:** When the body of mass 'm' is on the top of inclined plane, its potential energy is mgh. When it starts sliding without slipping its potential energy is converted into sliding kinetic energy.

P.E. = K.E.

$$mgh = \frac{1}{2} mv^2$$

Type - XXI

Numerical based on Rolling bodies

Formulae used

1. $v = \sqrt{\frac{2gh}{\left[1+\frac{K^2}{R^2}\right]}}$

2. $a = \frac{g\sin\theta}{1+k^2/R^2}$

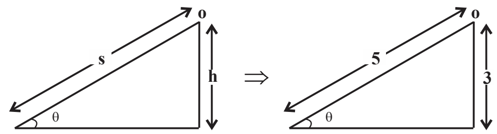
★ 1) Starting from rest, an object rolls down along an incline that rises by 3 in every 5 (along it). The object gains a speed of $\sqrt{10}$ m/s as it travels a distance of $\frac{5}{3}$ m along the incline. What can be the possible shape/s of the object?

Data: $v = \sqrt{10}$ m/s, u = 0 m/s

To find: Possible shape (s) of object.

Formula: $v^2 = \frac{2gh}{1+\frac{K^2}{R^2}}$

Solutions: An object rolls down along an incline that rises by 3 in every 5 means for travelling adistance of 5m, it height increases by 3 m.



From figure, $h = s \sin\theta$

where, $\sin\theta = \frac{3}{5}$ m and

$s = \frac{5}{3}$ m ....(given)

$\therefore \quad h = \frac{5}{3} \times \frac{3}{5} = 1$

From formula,

$$(\sqrt{10})^2 = \frac{2 \times 10 \times h}{1 + \frac{K^2}{R^2}}$$

$$\therefore \quad 10 = \frac{2 \times 10 \times 1}{1 + \frac{K^2}{R^2}}$$

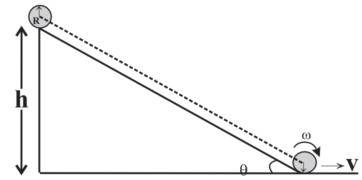
$$\therefore \quad 1 + \frac{K^2}{R^2} = 2$$

$\therefore \quad \frac{K^2}{R^2} = 1$, and it its possible for a ring or a hollow cylinder.

Ans : The possible shape of the object can be a ring or hollow cylinder.

2) Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are indentical. Which of the bodies reaches the ground with maximum velocity?

Solution: Suppose a body of mass m starting from rest rolls down an inclined plane. We assume there is no loss energy due to friction



Clearly the velocity v attained by the rolling body at the bottom of the inclined plane is independent of its mass

For a ring, $k^2 = R^2$

$$\therefore \quad v_{ring} = \sqrt{\frac{2gh}{1+1}} = \sqrt{gh}$$

For a solid cylinder,
$k^2 = R^2/2$

$$\therefore \quad v_{cylinder} = \sqrt{\frac{2gh}{1+1/2}} = \sqrt{\frac{4gh}{3}}$$

For a solid sphere,
$k^2 = 2R^2/5$

$\therefore$

Clearly among the three bodies the sphere has the greatest and the ring has the least velocity of the centre of mass at the bottom of the inclined plane.

Problem for Practice

1. A solid cylinder of mass 5 kg and radius 0.5 m rolls down an anclined plane of height 6m. Calculat its votational energy when at reaches the foot of the plane ($g = 10$ m/s^2)

   Ans : 85.7 J

2. A solid sphere of mass 5 kg rolls on a table with lenear speed 10 m/s find its total kinetic energy

   Ans : 350J

MULTIPLE CHOICE QUESTIONS
Entrance Corner (Set 11)

1. A solid cylinder of mass 2 kg and radius 50 cm rolls up an inclined plane of angle of inclination 30°. The centre of mass of the cylinder has speed of 4m/s. The distance travelled by the cylinder on the inclined surface will be, [take $g = 10$ m/s^2]

a. 2.4 m b. 2.2 m
c. 1.6 m d. 1.2 m

2. A solid sphere is in rolling motion. In rolling motion a body possesses translational kinetic energy (K_t) as well as rotational kinetic energy (K_r) simultaneously. The ratio $K_t : (K_t + K_r)$ for the sphere is
a. 7 : 10 b. 5 : 7
c. 2 : 5 d. 10 : 7

3. Three objects, A : (a solid sphere), B : (a thin circular disk) and C : (a circular ring), each have the same mass M and radius R. They all spin with the same angular speed their own symmetry axes. The amounts of work (W) required to bring them to rest, would satisfy the relation
a. $W_C > W_B > W_A$ b. $W_A > W_B > W_C$
c. $W_A > W_C > W_B$ d. $W_B > W_A > W_C$

4. A small object of uniform density rolls up a curved surface within initial velocity 'v'. It reaches upto a maximum height of respect to the initial position. The object is a
a. solid sphere b. hollow sphere
c. disc d. ring

5. A drum of radius R and mass M, rolls down without slipping along an inclined plane of angle . The frictional force
a. dissipates energy as heat
b. decreases the rotational motion
c. decreases the rotational energy and translational
d. converts translational energy to rotational energy

6. A spherical ball rolls on a table without slipping. Then the fraction of its total energy associated with rotaion is
a. 2/5 b. 2/7
c. 3/5 d. 3/7

7. A solid sphere, disc and solid cylinder all of the same mass and made of the same material are allowed to roll down (from rest) on the inclined plane, then
a. solid sphere reaches the bottom first
b. solid sphere reaches the bottom last
c. disc will reach the bottom first
d. all reach the bottom at the same time

8. The speed of a homogeneous solid sphere after rolling down aninclined plane of vertical height h from rest without sliding is
a. b.
c.

9. If a sphere is rolling, the ratio of the translational energy to total kinetic energy is given by
a. 7 : 10 b. 2 : 5
c. 10 : 7 d. 5 : 7

10. A solid cylinder of mass 2 kg and radius 0.2 m is rotating about its own axis without friction with angular velocity 3 rad/sec. A particle of mass 0.5 kg and moving with a velocity of 5 m/s strikes the cylinder and sticks to it as. The loss in energy due to collision will be-
a. 0.12 J b. 12 J
c. 1.2 J d. 1.12 J

Try yourself

11. A disc of radius 2 m and mass 100 kg rolls on a horizontal floor. Its centre of mass has speed of 20 cm/s. How much work is needed to stop it?
a. 3J b. 30kJ
c. 2J d. 1J

12. A disk and a sphere of same radius but different masses roll off on two inclined planes of the same altitude and length. Which one of the two objects gets to the bottom of the plane first?
a. Disk b. Sphere
c. Both reach at the same time
d. Depends on their masses

13. The ratio of the accelerations for solid sphere (mass 'm' and radius 'R') rolling down an

incline of angle ‘
slipping down the incline without rolling is :

a. 5 : 7 b. 2 : 3
b. 2 : 5 d. 7 : 5

14. A solid cylinder of mass m and radius R rolls down an inclined plane of height h without slipping. The speed of its centre of mass when it reaches the bottom is

a. $\sqrt{(2gh)}$ b. $\sqrt{4gh/3}$
c. $\sqrt{3gh/4}$ d. $\sqrt{4g/h}$

15. A solid cylinder and a hollow cylinder both of the same mass and same external diameter are released from the same height at the same time on an inclined plane, Both roll down without slipping. Which one will reach the bottom first?

a. Both together
b. Solid cylinder
c. One with higher density
d. Hollow cylinder

16. A thin uniform circular ring is rolling down an inclined plane of inclination 30° without slipping Its linear acceleration along the inclination plane will be

a. $\frac{g}{2}$ b. $\frac{g}{3}$
c. $\frac{g}{4}$ d. $\frac{2g}{3}$

17. A body is freely rolling down on an inclined plane whose angle of inclination is θ. If a is acceleration of its centre of mass then following is correct

a. $a = g\sin\theta$ b. $a < g\sin\theta$
c. $a > g\sin\theta$ d. $a = 0$

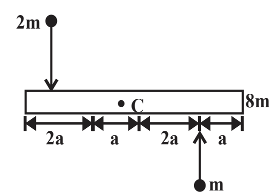
18. Solid sphere, solid cylinder, hollow sphere and hollow cylinder are simultaneously freely start rolling down from top of an inclined plane. If a is acceleration of their centre of mass then a is

a. maximum for solid cylinder, minimum for hollow sphere
b. maximum for hollow cylinder, minimum for solid sphere
c. maximum for hollow sphere, minimum for solid cylinder
d. maximum for solid sphere, minimum for hollow cylinder

19. A particle of mass 3 kg is moving under the action of a central froce whose potential energy is given by U (r) = 10 r^3 joule. For what energy and angular momentum will the orbit be a circle of radius 10m ?

a. 2.5×10^4J, 3000 kg m^2/sec
b. 2.5×10^3J, 3000 kg m^2/sec
c. 2.5×10^2J, 30000 kg m^2/sec
d. 2.5×10^2J, 300 kg m^2/sec

20. A uniform bar of length 6a and 1.2mass 8 m lies on a smooth horizontal table. Two point masses m and 2 m moving in the same horizontal plane with speeds 2v and v respectively strike the bar as shown in figure and stick the bar after collision. C represent centre of mass of bar. Denoting angular velocity (about the centre of mass,) total energy and centre of mass velocity by ω, E and v_c respectively, after collision we have,



a. $v_c = 0$
b. $w = 3v/5a$,
c. $w = v/5a$
d. $E = 3mv^2/5$

AnswerKey

Set - 1 (MCQ)

1	**d**	2	**b**	3	**c**	4	**a**	5	**c**
6	**c**	7	**b**	8	**b**	9	**b**	10	**c**

Try Yourself

11	**c**	12	**b**	13	**d**	14	**d**	15	**d**
16	**b**	17	**b**	18	**b**				

Set - 2 (MCQ)

1	**a**	2	**c**	3	**b**	4	**c**

Try Yourself

5	**a**	6	**a**	7	**d**	8	**c**

Set - 3 (MCQ)

1	c	2	b	3	a	4	c	5	a
6	b	7	a						

Try Yourself

8	a	9	d	10	b	11	b	12	d
13	c	14	b	15	a				

Set - 4 (MCQ)

1	b	2	d	3	a	4	a	5	c
6	a	7	d	8	c	9	d	10	b

Try Yourself

11	a	12	c	13	a	14	b	15	d
16	c	17	d	18	b	19	b	20	c

Set - 5 (MCQ)

1	d	2	b	3	c	4	d

Try Yourself

5	c	6	c	7	d	8	d

Set - 6 (MCQ)

1	a	2	a	3	c	4	d	5	c
6	c	7	c	8	b				

Try Yourself

9	a	10	d	11	c	12	a	13	c
14	c	15	d	16	b				

Set - 7 (MCQ)

1	a	2	a	3	d	4	c	5	d
6	b	7	d	8	c	9	d	10	b

Try Yourself

11	b	12	b	13	b	14	c	15	b
16	c	17	a	18	c	19	d	20	d

Set - 8 (MCQ)

1	b	2	c	3	c	4	c	5	c

Try Yourself

6	b	7	a	8	a	9	a	10	d

Set - 9 (MCQ)

1	c	2	b	3	a	4	a	5	a
6	c	7	d	8	d	9	c		

Try Yourself

10	b	11	c	12	d	13	b	14	a
15	a								

Set - 10 (MCQ)

1	b	2	c	3	c	4	d	5	c
6	c	7	b	8	d	9	a	10	b

Try Yourself

11	b	12	c	13	b	14	c	15	d
16	b	17	b	18	d	19	d	20	a

Set - 11 (MCQ)

1	a	2	b	3	a	4	c	5	d
6	b	7	a	8	a	9	d	10	a

Try Yourself

11	a	12	b	13	a	14	b	15	b
16	c	17	b	18	b	19	c	20	d

❑❑❑